Hereditary Properties of Graphs

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First problem

Definition

The detour order (of a vertex v) of a graph G is the order of a longest path (beginning at v).
The detour sequence of G is a sequence consisting of the detour orders of its vertices.
A graph is called a detour graph if its detour sequence is constant.

Problem

Find **necessary and sufficient conditions** *for a given sequence of positive integers to be the detour sequence of some graph.*

Two well-known conjectures

$\tau(G)$ **- detour order of** G

PPC - Path Partition Conjecture

For any graph G and arbitrary positive integers a, b satisfying $a + b = \tau(G)$, **the vertices of a graph** G **can be partitioned into two parts** A **and** B in such a way that **the order of a longest path in** $G[A]$ **is at most** a and **the order of a longest path in** $G[B]$ **is at most** b.

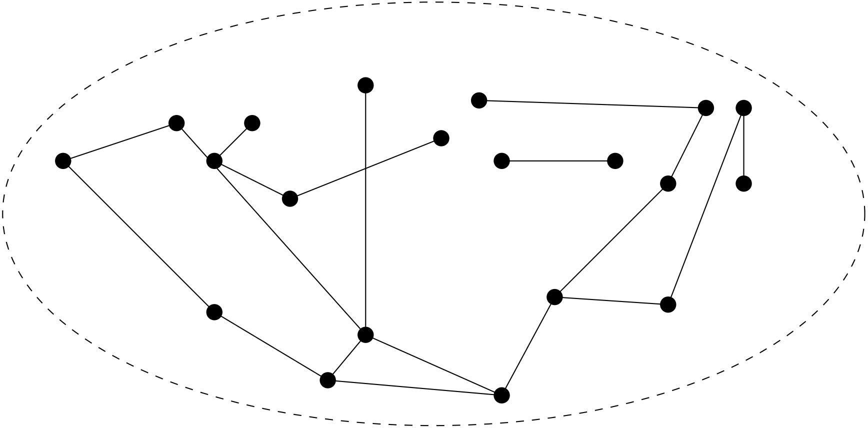
A subset K of V is called a P_n- kernel of G if $\tau(G[K]) \leq n - 1$ and **every vertex** $v \in V \setminus K$ **is adjacent to an end-vertex of a path of order** $n - 1$ **in** $G[K]$.

PKC - Path Kernel Conjecture

Every graph has a P_k-*kernel* for every positive integer $k \geq 2$.

Generalised colouring of graphs

Given a graph G and properties $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$. **We would like to assign colours to the vertices of G according to this prescription.**



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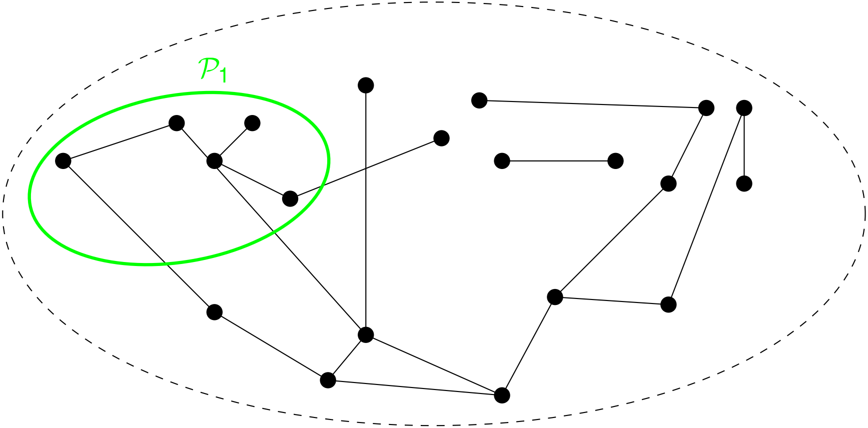

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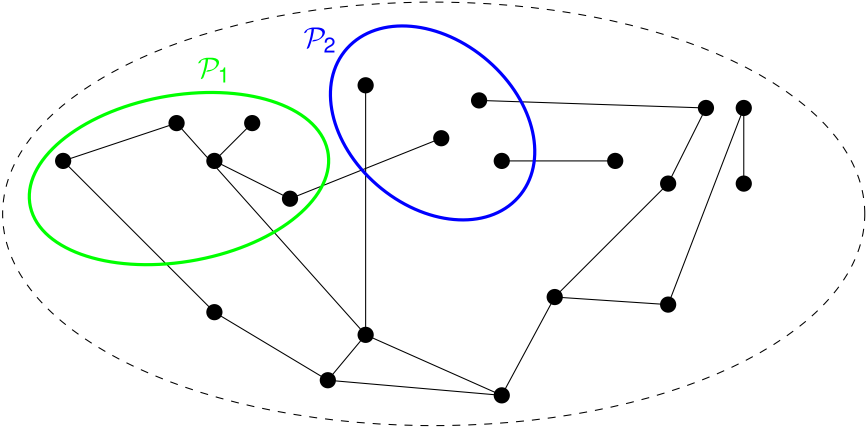

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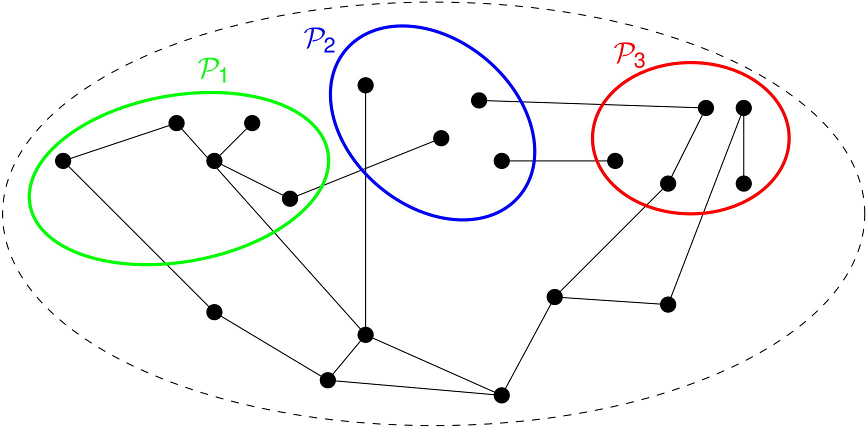

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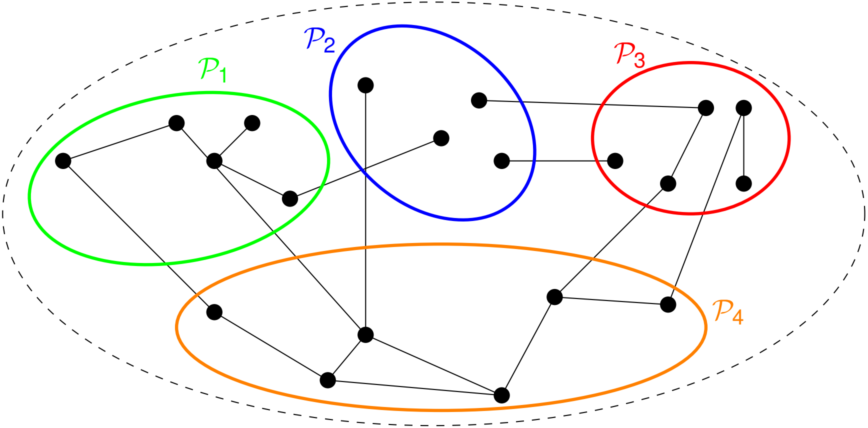

Properties of graphs

Definition

Let $\mathcal{I}^\omega$ and $\mathcal{I}$ denote the class of all simple countable graphs, simple finite graphs respectively. A graph property $\mathcal{P}$ is any isomorphism-closed nonempty subclass of $\mathcal{I}^\omega$ ($\mathcal{I}$).

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Definition

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties. A vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colouring (partition) of a graph $G = (V, E)$ is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that each colour class V_i induces a subgraph $G[V_i]$ having property $\mathcal{P}_i$.

If each of the $\mathcal{P}_i$'s, $i = 1, 2, \ldots, n$, is the property $\mathcal{O}$ of being **edgeless**, we have the well-known proper vertex n-colouring.

Reducible properties of graphs

Notation

Let us denote by $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n,\ n \geq 2$ the set of all $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable graphs.

The binary operation $\circ$ is obviously commutative, associative on the class of graph properties and $\mathcal{E} = \{K_0\}$ is its neutral element.

Definition

A nontrivial graph property $\mathcal{P}$ is said to be reducible if there exist nontrivial graph properties $\mathcal{P}_1, \mathcal{P}_2$, such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$; otherwise $\mathcal{P}$ is called irreducible.

Problem

What properties are reducible?

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Hereditary properties

Definition

Let $\preceq$ be a partial order on the set $\mathcal{I}$. Then we say that a property $\mathcal{P} \subseteq \mathcal{I}$ is $\preceq$-hereditary if from the facts $G \in \mathcal{P}$ and $H \preceq G$ follows that $H \in \mathcal{P}$.
A property $\mathcal{P}$ is said to be induced-hereditary if it is closed under taking induced subgraphs.
A property $\mathcal{P}$ is said to be hereditary if it is closed under taking subgraphs.

Special properties of graphs

Definition

Let $\mathcal{P}$ be a graph property, $\mathcal{P}$ is of finite character if a graph in $\mathcal{I}^{\omega}$ has property $\mathcal{P}$ if and only if each its finite induced subgraph has property $\mathcal{P}$.

It is easy to see that if $\mathcal{P}$ is of finite character and a graph has property $\mathcal{P}$ then so does every induced subgraph.

Special properties of graphs

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Thus properties of finite character are induced-hereditary. However not all induced-hereditary properties are of finite character

Induced-hereditary properties of graphs

Example

The properties

- to be edgeless,
- to be of maximum degree at most k,
- to be K_n-free,
- to be acyclic,
- to be complete,
- to be perfect

are properties of finite character.

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Additivity

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A property of graphs is called additive if it is closed with respect to taking disjoint union.

Example

The property to be acyclic is additive however the property to be complete it is not.

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Vertex Colouring Compactness Theorem

Theorem (Cowen, Hechler, Mihók, 2002)

Let G be a graph in $\mathcal{I}^{\omega}$ *and let* $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ *be properties of graphs of finite character. Then G is* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-colourable if and only if every finite induced subgraph of G is* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-colourable.*

It provides a generalisation of the classical result of de Bruijn and Erdős who proved that an infinite graph G is k-colourable if and only if every finite subgraph of G is k-colourable.

Lattice of hereditary properties of graphs

Theorem (Borowiecki and Mihók, 1991)

Partially ordered sets $\mathbb{L}_{\subseteq} = (\mathbb{L}, \subseteq)$ *and* $\mathbb{L}^a_{\subseteq} = (\mathbb{L}^a, \subseteq)$ *form complete, distributive lattice with the least element* $\mathcal{E}$ *and the greatest element* $\mathcal{I}$.

Analogous results can be proved for the sets $\mathbb{L}_{\leq} = (\mathbb{L}, \leq)$ and $\mathbb{L}^a_{\leq} = (\mathbb{L}^a, \leq)$.

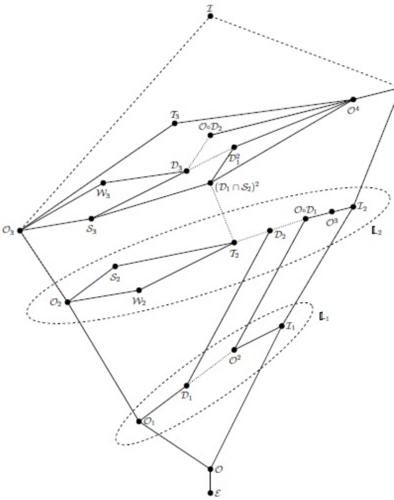
Lattice of hereditary properties of graphs - cont.

If α is any cardinal, then $\mathbb{L}(\alpha)$ denotes the set of all hereditary properties that contains graphs with vertex set of order at most α.

Theorem (Jakubík, 2002)

Partially ordered sets $\mathbb{L}(\alpha)^a_{\subseteq} = (\mathbb{L}^a(\alpha), \subseteq)$ *and* $\mathbb{L}(\alpha)^a_{\leq} = (\mathbb{L}(\alpha)^a, \leq)$ *form completely distributive lattice with the least element* $\mathcal{E}$ *and the greatest element* $\mathcal{I}$. $\mathcal{I}$.

Three sisters



A characterisation in terms of forbidden substructures

Let $\mathcal{P} \subseteq \mathcal{I}$. We say that $\mathcal{P}$ can be characterized in terms of forbidden subgraphs if there exists a subset $\mathbf{F}(\mathcal{P})$ of $\mathcal{I}$ such that $G \in \mathcal{P}$ if and only if no subgraph of G is in $\mathbf{F}(\mathcal{P})$.

Theorem (Greenwell, Hemminger and Klerlein, 1973)

Let $\mathcal{I}$ be a partially ordered class of graphs under $\subseteq$. A property $\mathcal{P}$ can be characterized in terms of forbidden subgraphs if and only if for all $G \in \mathcal{P}$, $H \subseteq G$ implies $H \in \mathcal{P}$.

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Minimal forbidden graphs

It is easy to see that if $F, F' \in \mathbf{F}(\mathcal{P})$, $F \subseteq F'$ and we put $\boldsymbol{F'}(\mathcal{P}) = \mathbf{F}(\mathcal{P}) \setminus \{F'\}$, then $\mathcal{P}$ is uniquely determined by $\boldsymbol{F'}(\mathcal{P})$, too.

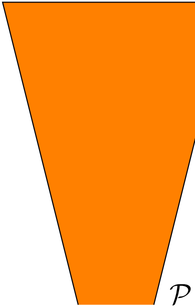
Definition

The set of minimal forbidden subgraphs of $\mathcal{P}$ is the set:

$$\mathbf{F}(\mathcal{P}) = \left\{G \in \mathcal{I} \setminus \mathcal{P} : \text{ each proper subgraph of } G \text{ belongs to } \mathcal{P}\right\}.$$

Note that $\mathbf{F}(\mathcal{P})$ may be finite or infinite. A property is additive if and only if $\mathbf{F}(\mathcal{P})$ contains only connected graphs.

Minimal forbidden graphs



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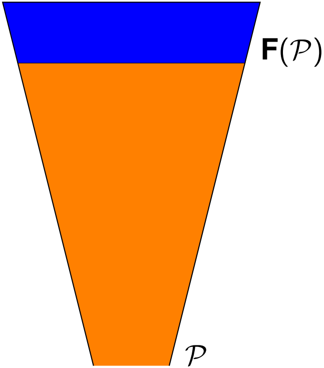

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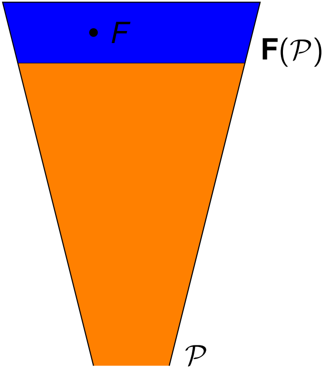

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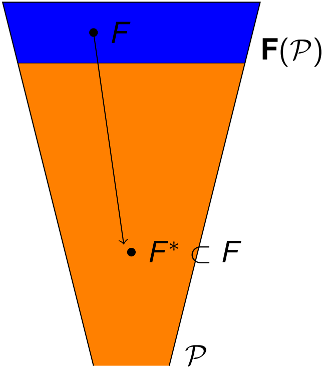


Examples of minimal forbidden graphs

Example

Let S_n and P_n denote the star and the path on n vertices, respectively. Then

- $\mathbf{F}(\mathcal{O}) = \{K_2\}$;
- $\mathbf{F}(\mathcal{O}_k) = \{H \in \mathcal{I} : H \text{ is a tree on } k+2 \text{ vertices}\}$,
- $\mathbf{F}(\mathcal{S}_k) = \{S_{k+2}\}$;
- $\mathbf{F}(\mathcal{W}_k) = \{P_{k+2}\}$;
- $\mathbf{F}(\mathcal{T}_k) = \{H \in \mathcal{I} : H \text{ is homeomorphic to } K_{k+2} \text{ or } K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\}$;
- $\mathbf{F}(\mathcal{I}_k) = \{K_{k+2}\}$.

Examples of minimal forbidden graphs (2)

For a non-negative integer k and a graph G, we denote the set of all vertices of G of degree $k+1$ by $M(G)$. If $S \subseteq V(G)$ is a cutset of vertices of G and $G_1, G_2, \ldots, G_s$, $s \geq 2$ are the components of $G - S$, then the graph $G - V(G_i)$ is denoted by H_i, $i = 1, 2, \ldots, s$

Theorem (Mihók, 1981)

A graph G belongs to $\mathbf{F}(\mathcal{D}_k)$ *if and only if*

1. *G is connected,*
2. $\delta(G) \geq k + 1$,
3. $V(G) \setminus M(G)$ *is an independent set of vertices of G*
4. *and for each cutset* $S \subset V(G) \setminus M(G)$ *we have that* $\delta(H_i) \leq k$ *for each* $i = 1, 2, \ldots, s$.

An extension of Brook's theorem

Theorem (Mihók, 1992)

Let T be a tree on $k+2$ vertices, $k \geq 3$. If a connected graph G does not contain T, then G is k-colorable **unless** $G = K_{k+1}$.

Maximal graphs

Any monotone property is uniquely determined by its set of minimal forbidden subgraphs. An alternative way is to characterize $\mathcal{P}$ by the set of graphs containing all the graphs in $\mathcal{P}$ as subgraphs.

Definition

To be more accurate, let us define the set of $\mathcal{P}$-maximal graphs by

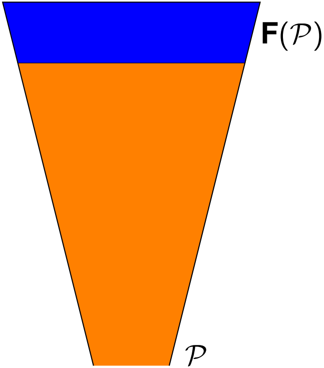
$$\mathbf{M}(\mathcal{P}) = \{G \in \mathcal{P} : G + e \notin \mathcal{P} \text{ for each } e \in E(\overline{G})\}$$

and the set of $\mathcal{P}$-maximal graphs of order n by

$$\mathbf{M}(\mathcal{P}, n) = \{G \in \mathcal{P} : |V(G)| = n \text{ and } G \in \mathbf{M}(\mathcal{P})\}.$$

From these definitions it is evident that $\mathbf{M}(\mathcal{P}) = \bigcup_{n \geq 1} \mathbf{M}(\mathcal{P}, n)$.

Maximal graphs



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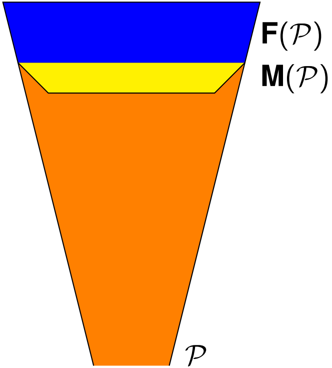

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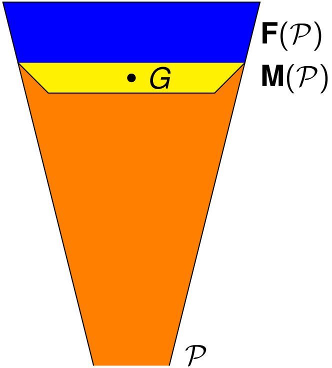

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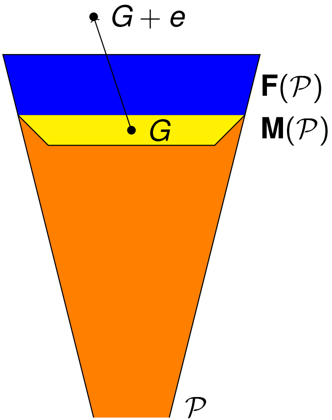

Examples of maximal graphs

Example

- $\mathbf{M}(\mathcal{O}^2) = \{G \in \mathcal{I} : G$ is complete bipartite$\}$;
- $\mathbf{M}(\mathcal{S}_k) = \{G \in \mathcal{I} : G$ is almost $k+1$-regular$\}$;
- $\mathbf{M}(\mathcal{D}_1) = \{G \in \mathcal{I} : G$ is a tree$\}$;
- $\mathbf{M}(\mathcal{T}_3) = \{G \in \mathcal{I} : G$ is a triangulation of the plane$\}$;
- $\mathbf{M}(\mathcal{PT}_k) = \{G \in \mathcal{I} : G$ is a partial k-tree$\}$;
- $\mathbf{M}(\mathcal{D}_k) = \{G \in \mathcal{I} : G$ is maximal k-degenerate$\}$;
- $\mathbf{M}(\mathcal{O}_k) = \{G \in \mathcal{I} : G = K_{i_1} \cup K_{i_2} \cup \cdots \cup K_{i_j}$ s.t. for each $r, s = 1, 2, \ldots, j, r \neq s$ holds $i_r \leq k+1$ and $i_r + i_s \geq k+2\}$.

Some relationships

Lemma (Borowiecki and Mihók, 1991)

Let $\mathcal{P}_1, \mathcal{P}_2$ be any hereditary properties. Then the following statements are mutually equivalent:

1. $\mathcal{P}_1 \subseteq \mathcal{P}_2$;
2. *for each* $H \in \mathbf{F}(\mathcal{P}_2)$ *there exists* $H' \in \mathbf{F}(\mathcal{P}_1)$ *such that* $H' \subseteq H$;
3. *for any positive integer* n *and an arbitrary* $G \in \mathbf{M}(n, \mathcal{P}_1)$ *there is* $G' \in \mathbf{M}(n, \mathcal{P}_2)$ *such that* $G \subseteq G'$.

Generating sets

It is natural that the characterization of hereditary property in terms of $\mathcal{P}$-maximal graphs leads to a more general concept — the generating set of a hereditary (an additive hereditary) property.

Consider an arbitrary set $\mathcal{G}$, a subset of $\mathcal{I}$. It is quite easy to see that the property

$$[\mathcal{G}] = \{G \in \mathcal{I} : G \text{ is a subgraph of some graph } H \in \mathcal{G}\}$$

is hereditary, and the property

$$[\mathcal{G}]^a = \{G \in \mathcal{I} : \text{ each component of } G \text{ is a subgraph of some } H \in \mathcal{G}\}$$

is in addition additive and both are generated by the set $\mathcal{G}$, called generating set.

Generators

Definition

A graph G (finite or infinite) is called strong generator of a property $\mathcal{P}$ if $[\{G\}] = \mathcal{P}$.

Chains of hereditary properties

Definition

For a normalised invariant φ we define properties

$$\mathcal{P}_{(\varphi,k)} = \{G \in \mathcal{I} : \varphi(G) \leq k\}, k = 0, 1, \ldots .$$

Then the chain

$$\mathcal{P}_{(\varphi,0)} \subset \mathcal{P}_{(\varphi,1)} \subset \cdots \subset \mathcal{P}_{(\varphi,n)} \subset \ldots$$

of hereditary properties is called the chain associated with (a normalised graph invariant) φ.

Invariants derived form chains

Suppose that in the lattice $\mathbb{L}_{\subseteq}$ we have a finite or countable chain

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_k \subset \cdots, \quad k \geq 0.$$

Let

$$\mathcal{P} = \bigcup_{k \geq 0} \mathcal{P}_k.$$

Then an invariant $f \colon \mathcal{P} \to \mathbb{N}$ by this chain is defined as follows:

- $f(G) = 0$, if $G \in \mathcal{P}_0$,
- $f(G) = k$, if $G \in \mathcal{P}_k \setminus \mathcal{P}_{k-1}$, for $k \geq 1$.

Extremal graphs

Problem

Given a family $\mathbf{F}(\mathcal{P})$ *of forbidden subgraphs, find the number*

$$ex(n, \mathcal{P}) = \max\{|E(G)| : G \in \mathbf{M}(n, \mathcal{P})\}.$$

Definition

The $\mathcal{P}$-maximal graphs of order n with exactly $\mathrm{ex}(n, \mathcal{P})$ edges are called $\mathcal{P}$*-extremal* graphs.

A generalisation of Turan's theorem

Theorem (Erdős and Stone, 1966)

If $\mathcal{P}$ is a hereditary property with chromatic number $\chi(\mathcal{P}) = \min\{\chi(F) : F \in \mathbf{F}(\mathcal{P})\}$, then

$$\mathrm{ex}(n, \mathcal{P}) = \left(1 - \frac{1}{\chi(\mathcal{P}) - 1}\right)\binom{n}{2} + o(n^2).$$

The meaning of this theorem is that $\mathrm{ex}(n, \mathcal{P})$ depends only very loosely on the structure of $\mathcal{P}$, it is already determined by $\chi(\mathcal{P})$. By the above theorem $\mathrm{ex}(n, \mathcal{P}) = o(n^2)$ if and only if $\mathcal{P}$ is a degenerate property, i.e. its chromatic number is 2.

Saturated graphs

It is natural to investigate also the “opposite side”.

Definition

$$\mathrm{sat}(n, \mathcal{P}) = \min\{|E(G)| : G \in \mathbf{M}(n, \mathcal{P})\}.$$

The graphs of order n with $\mathrm{sat}(n, \mathcal{P})$ edges are called $\mathcal{P}$-saturated.

Saturated graphs

Theorem (Kászonyi and Tuza, 1986)

If $\mathcal{P}$ is a given hereditary property and

1. $u = u(\mathcal{P}) = \min\Big\{|V(F)| - \beta(F) - 1 : F \in \mathbf{F}(\mathcal{P})\Big\}$,
2. $d = d(\mathcal{P}) = \min\Big\{|E(F')| : F' \subseteq F \in \mathbf{F}(\mathcal{P})$ *is induced by a set* $S \cup \{x\}, S \subseteq V(F)$ *is independent,* $x \in V(F) \setminus S$ *and* $S| = |V(F)| - u - 1\Big\}$,

then

$$\mathrm{sat}(n, \mathcal{P}) \leq un + \frac{1}{2}(d-1)(n-u) - \binom{u+1}{2},$$

if n is large enough.

An intriguing problem

Problem

For what hereditary properties is true that:

$$\mathrm{ext}(n, \mathcal{P}) = \mathrm{sat}(n, \mathcal{P})?$$

Some positive answers:

- planar graphs
- outerplanar graphs
- k-degenerate graphs.

Minimal reducible bounds - motivation

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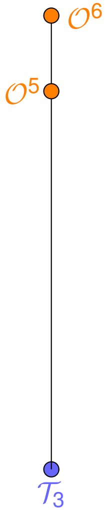

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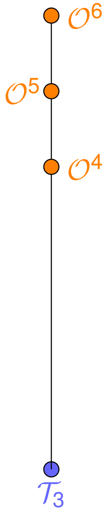

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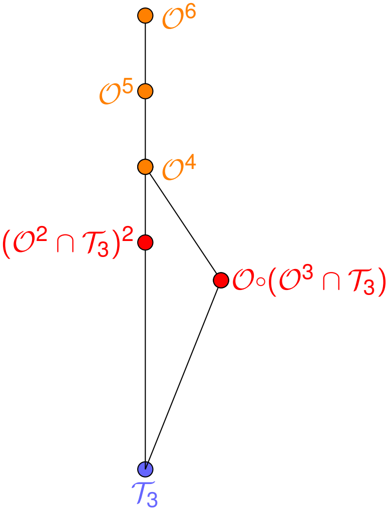

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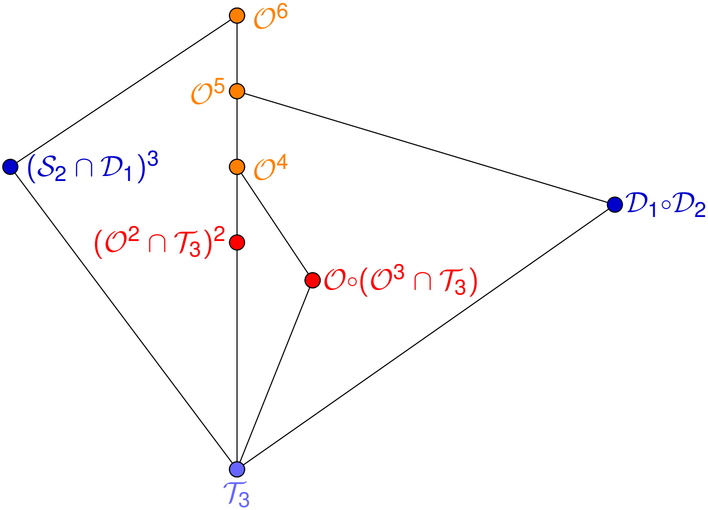
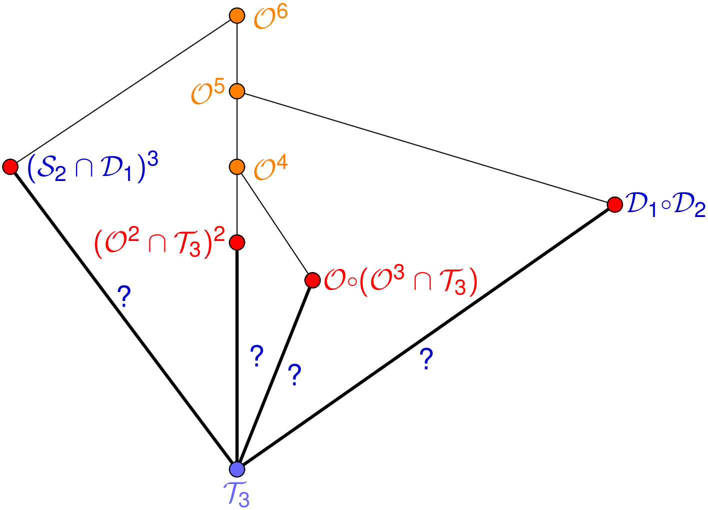


Minimal reducible bounds - motivation



Minimal reducible bound

Definition

A reducible property $\mathcal{R} \in \mathbb{L}^{a}_{\leq}$ ($\in \mathbb{L}^{a}_{\subseteq}$) is called a minimal reducible bound for a property $\mathcal{P} \in \mathbb{L}^{a}_{\leq}$ ($\in \mathbb{L}^{a}_{\subseteq}$) if $\mathcal{P} \subseteq \mathcal{R}$ and there is no reducible property $\mathcal{R}^* \in \mathbb{L}^{a}_{\leq}$ ($\in \mathbb{L}^{a}_{\subseteq}$) satisfying $\mathcal{P} \subseteq \mathcal{R}^* \subset \mathcal{R}$.

This means that the obtained colouring result is sharp and in some sense cannot be improved.
The problem of finding all minimal reducible bounds for the class of planar graphs was formulated by Mihók and Toft in 1993 (Problem 17.9)

On the existence of minimal reducible bounds

Theorem (Berger, 2001)

- *Every additive hereditary property has at least one minimal reducible bound.*
- *All the reducible bounds of a property $\mathcal{P}$ contain a minimal reducible bound for $\mathcal{P}$.*
- *Every reducible additive hereditary property is a minimal reducible bound for some irreducible additive hereditary property.*

Theorem

Let $\mathcal{P}$ be an additive induced-hereditary (hereditary) property of graphs with $\chi(\mathcal{P}) = k$. Then all the minimal reducible bounds for $\mathcal{P}$ are properties consisting of at most k irreducible factors.

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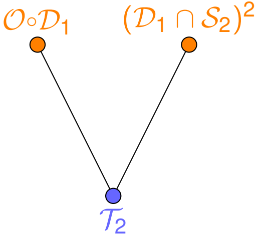
Two interesting results



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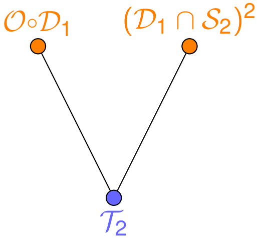

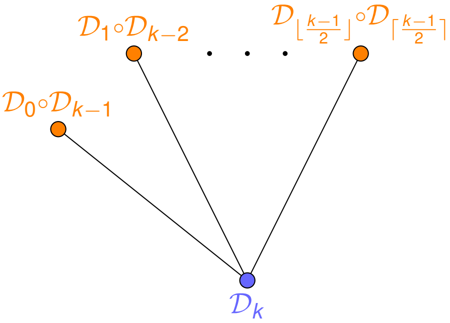

Some results

Theorem

- $\mathbf{B}_{\mathbf{L}_{\subseteq}}(\mathcal{O}_k) = \{\mathcal{O}_p \circ \mathcal{O}_q : p + q + 1 = k\}$;
- $\mathbf{B}_{\mathbf{L}_{\subseteq}}(\mathcal{D}_k) = \{\mathcal{D}_p \circ \mathcal{D}_q : p + q + 1 = k\}$;
- $\mathbf{B}_{\mathbf{L}_{\subseteq}}(\mathcal{P}T_k) = \{\mathcal{P}T_p \circ \mathcal{P}T_q : p + q + 1 = k\}$;
- $\mathbf{B}_{\mathbf{L}_{\subseteq}}(\mathcal{T}_2) = \{\mathcal{O} \circ \mathcal{D}_1, (\mathcal{D}_1 \cap \mathcal{S}_2)^2\}$;
- $\mathbf{B}_{\mathbf{L}_{\leq}}(\mathcal{D}_k) = \{\mathcal{D}_p \circ \mathcal{D}_q : p + q + 1 = k\}$.

A general criteria

Theorem

Let $\mathcal{O} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_k$, $k \geq 1$ be a chain of additive induced-hereditary degenerate properties of graphs. If for arbitrary non-negative integers r, s, t, u, $r + s + 1 = k$, $t + u = k$ the properties $\mathcal{P}_r, \mathcal{P}_s, \mathcal{P}_t, \mathcal{P}_u$ satisfy the following two conditions

(i) $\mathcal{P}_k \subseteq \mathcal{P}_r \circ \mathcal{P}_s$;

(ii) $\mathcal{P}_k \overset{i}{\to} (\mathcal{P}_t, \mathcal{P}_u)$,

then the set of minimal reducible bounds for $\mathcal{P}_k$ in the lattice $\mathbb{L}^a_{\leq}$ is of the form $\mathbf{B}_{\mathbb{L}_{\leq}}(\mathcal{P}_k) = \{\mathcal{P}_p \circ \mathcal{P}_q : p + q + 1 = k\}$.

A characterisation of hereditary properties is difficult

- A. Berger proved that any **reducible additive hereditary property** of graphs **has infinitely many minimal forbidden graphs**.
- only very little is known about **the structure of F**$(\mathcal{P}\circ\mathcal{Q})$, even in the case when **the structure of F**$(\mathcal{P})$ **and F**$(\mathcal{Q})$ **is known**

Useful information on the structure of **F**$(\mathcal{P}\circ\mathcal{Q})$ can be obtained by investigation of **graph invariants associated with a property** $\mathcal{P}\circ\mathcal{Q}$**.**

Monotone invariants

By a graph invariant φ we mean any integer-valued (real-valued) function defined on $\mathcal{I}$ such that $\varphi(G) = \varphi(H)$ for each pair G, H of isomorphic graphs.

Definition

We say that the invariant φ interpolates over the class $\mathcal{P}$ of graphs if for any $G, H \in \mathcal{P}$ and each integer k between $\varphi(G)$ and $\varphi(H)$ there exists a graph $F \in \mathcal{P}$ such that $\varphi(F) = k$. A graph invariant φ is called monotone if $G \subseteq H$ implies $\varphi(G) \leq \varphi(H)$. A graph invariant φ is called additive if $\varphi(G \cup H) \leq \max\{\varphi(G), \varphi(H)\}$ for any graphs G, H.

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ARHP invariants

Given a graph invariant φ, we define the associated invariant of a property $\mathcal{P}$ in the following manner:

$$\varphi(\mathcal{P}) = \min\{\varphi(F) : F \in \mathbf{F}(\mathcal{P})\}.$$

Definition

We say that a graph invariant φ is additive with respect to reducible hereditary properties (abbreviated by **ARHP**) if for any reducible property $\mathcal{P}\circ\mathcal{Q}$ the equality $\varphi(\mathcal{P}\circ\mathcal{Q}) = \varphi(\mathcal{P}) + \varphi(\mathcal{Q})$ is valid.

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A relationship between maximal and forbidden graphs

A graph invariant $\varphi(\mathcal{P})$ strongly depends on the features of the minimal forbidden subgraphs.

Lemma

Let φ be a monotone graph invariant such that, for every graph G and every edge e from its complement, $\varphi(G + e) \leq \varphi(G) + 1$ and let $\mathcal{P}$ be a hereditary graph property. Then for any graph $G \in \boldsymbol{M}(n, \mathcal{P})$ with $n \geq c(\mathcal{P}) + 2$ the following holds: $\varphi(G) \geq \varphi(\mathcal{P}) - 1$.

A characterisation of ARHP invariants

For a graph invariant φ we can define an associated graph invariant $\hat{\varphi}$ in the following way:

$$\hat{\varphi}(G) = \max_{H \subseteq G} \varphi(H).$$

Lemma

If for some invariant φ there is an constant c, such that the equality $\varphi(\mathcal{Q}_1 \circ \mathcal{Q}_2) = \varphi(\mathcal{Q}_1) + \varphi(\mathcal{Q}_2) + c$ holds for all hereditary properties $\mathcal{Q}_1$ and $\mathcal{Q}_2$, then the invariant $\varphi^*(G) = \varphi(G) + c$ is ARHP.

A characterisation of ARHP invariants - cont.

Theorem

Let φ be a normalised graph invariant and $\mathcal{P}_{(\varphi,0)} \subset \mathcal{P}_{(\varphi,1)} \subset \dots$ be the chain associated with φ. Then φ is additive with respect to hereditary properties if and only if for every pair of non-negative integers k, l the following conditions hold:

(i) $\mathcal{P}_{(\varphi,k+l+1)} \subseteq \mathcal{P}_{(\varphi,k)} \circ \mathcal{P}_{(\varphi,l)}$;

(ii) $\mathcal{P}_{(\varphi,k+l)} \rightarrow (\mathcal{P}_{(\varphi,k)}, \mathcal{P}_{(\varphi,l)})$.

Examples of ARHP invariants

Example

The following invariants are ARHP

- $p^*(G) = p(G) - 1$, where $p(G)$ denotes the order of a graph G;
- $\pi^*(G)$, where $\pi(G)$ is the size of the largest tree contained in a graph G;
- the subchromatic number $\psi(G) = \chi(G) - 1$;
- the degeneracy number $\hat{\delta}(G) = col(G) - 1$ (sometimes denoted also by $\rho(G)$);
- the tree-width tw

Conjecture on sub-choice number

It is well known, that for any graph G we have

$$\chi(G) \leq ch(G) \leq col(G) \leq \Delta(G) + 1.$$

We proved that the subchromatic number ψ and the degeneracy number $\hat{\delta}(G)$ are ARHP. We conjecture that the 'sub-choice number' $ch^*(G) = ch(G) - 1$ is ARHP, too.

Conjecture

The sub-choice number $ch^*(\mathcal{P}) = ch(G) - 1$ is ARHP.

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Unique Factorisation Problem

Problem

Given a reducible additive property of finite character $\mathcal{R}$. Does there exist a factorisation of $\mathcal{R}$ into finite number of irreducible factors? Is the factorisation unique?

This problem states a natural question that provides an analogue of Fundamental Theorem of Arithmetics.

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History of UFT

- 1993 - Mihók, G.S. - UFT for properties without K_4
- 1995 - Mihók, Vasky - UFT for properties without K_5
- 1996 - Kratochvíl, P.M. - UFT for hom-properties
- 1997 - Mihók, G.S., Vasky - UFT for additive hereditary properties
- 2000 - Mihók - UFT for additive induced-hereditary properties
- 2003 - Farrugia, Mihók, Richter, G.S. - UFT from compositive induced-hereditary properties
- 2003 - Imrich, Mihók, G.S. - UFT for properties of finite character

What is the benefit from an application of FCA?

The main problem of the proof of UFT is the construction of irreducible factors. This process is rather technical and complicated and the proof of the uniqueness requires another non-trivial effort.

Observation

FCA allows us to identify and describe the irreducible factors in very natural way and therefore it can help to understand the proof and provide a suggestion how to find the factors algorithmically.

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Context for graph properties

Definition

Let us define a context (O, M, I) by setting objects to countable simple graphs, e.g. $O = \mathcal{I}^{\omega}$. For each connected finite simple graph $F \in \mathcal{I}$ let us consider an attribute m_F : do not contain an induced-subgraph isomorphic to F.

Thus GIm_F means that the graph G does not contain any induced subgraph isomorphic to F.

For an object $g \in O$ we write $g' = \{m \in M | gIm\}$ and γg for the **object concept** (g'', g'), where $g'' = \{\{g\}'\}'$.

An illustration

Example

- If G is planar then GIm_{K_5}.
- If G is acyclic then GIm_{C_n}, for any positive $n \geq 3$, where C_n denotes the cycle of order n.
- For any tree T of order n, it is not valid K_nIm_T because K_n contains all trees of order at most n as a subgraph.

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UFT for properties of finite character

Theorem

Every reducible additive property $\mathcal{R}$ of finite character is uniquely factorisable into finite number of irreducible factors belonging to $\mathbb{M}^{\omega a}$.

Remark

The assumption of additivity is substantial.

Uniquely partitionable graphs

Theorem

Let $\mathcal{P}$ be a hereditary property and n be a positive integer. Then a uniquely $\mathcal{P}^n$-partitionable graph exists if and only if $\mathcal{P}$ is irreducible.

A generalisation of UFT

One can verify that for the used method of the proof it is not important that we are dealing with simple graphs. Therefore, the proof can be applied for directed graphs, hypergraphs or partially ordered sets, i.e. relational structures.

Definition

A concrete category **C** is a collection of objects and arrows called morphisms. An object in a concrete category **C** is a set with structure. The morphism between two objects is considered to be a structure preserving mapping.

The natural examples of concrete categories are: **Set** of sets, **FinSet** of finite sets, **Graph** of graphs, **Poset** of partially ordered sets with structure preserving mappings, i.e. the homomorphisms of corresponding structures.

UFT for system of objects

Theorem

Every reducible additive property $\mathcal{R}$ of object-systems of finite character is uniquely factorisable into finite number of irreducible factors.

Thank you very much for your attention