Independence systems

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Coloring of a graph

Definition

A coloring of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color.

Generalization

The set of all vertices with any one color

- induces a subgraph without edges – hereditary properties
- is an independent set – independence systems

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Independence systems

Definition

An independence system is a pair $I = (X, \mathcal{I})$, where X is a finite set and $\mathcal{I}$ is a collection of subsets of X closed under inclusion, i.e.,

$$\text{if } A \in \mathcal{I} \text{ and } B \subseteq A, \text{ then } B \in \mathcal{I}.$$

Independence systems

A set $A \subseteq X$ is called

- independent set if $A \in \mathcal{I}$;
- base if it is a maximal independent set $(A \in \mathcal{B}(\mathcal{I}))$;
- dependent set if $A \notin \mathcal{I}$;
- circuit if it is a minimal dependent set $(A \in \mathcal{C}(\mathcal{I}))$.

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Independence systems

Determinations

- $I = (X, \mathcal{I})$, where $(A \subseteq B, B \in \mathcal{I}) \Rightarrow A \in \mathcal{I}$;
- $I = (X, \mathcal{B})$, where $(A, B \in \mathcal{B}) \Rightarrow A \not\subseteq B, B \not\subseteq A$;
- $I = (X, \overline{\mathcal{I}})$, where $(B \subseteq A, B \in \overline{\mathcal{I}}) \Rightarrow A \in \overline{\mathcal{I}}$;
- $I = (X, \mathcal{C})$, where $(A, B \in \mathcal{C}) \Rightarrow A \not\subseteq B, B \not\subseteq A$.

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Connection

Let P be a hereditary property of graphs. If G is a graph with the vertex set $V(G)$, then $I_P(G) = \big(V(G), \mathcal{I}_P(G)\big)$, where

$$\mathcal{I}_P(G) = \{A \subseteq V(G) : \langle A \rangle_G \in P\},$$

is an independence system.

Transversal

If $I = (X, \mathcal{I})$ is an independence system, then the collection

$$\mathcal{I}^* = \{A \subseteq X : X - A \notin \mathcal{I}\}$$

is closed under the set inclusion.

- $I^* = (X, \mathcal{I}^*)$ is an independence system.
- $(I^*)^* = I$

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Transversal

Let $I = (X, \mathcal{I})$ be an independence system.

- $\alpha(I)$ the number of elements in a smallest dependent set.
- $\beta(I)$ the number of elements in the largest independent set.

Theorem

Let $I = (X, \mathcal{I})$ be an independence system. Then $\alpha(I^*) + \beta(I) = |X|$ and $\alpha(I) + \beta(I^*) = |X|$.

Gallai

For every graph G of order n, $\alpha_0(G) + \beta_0(G) = n$.

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Partitions

Definition

Let $I = (X, \mathcal{I})$ be an independence system. A partition $\{X_1, \dots, X_t\}$ of X is called

- $\mathcal{I}$-partition of X if X_i is an independent set for each i;
- $\mathcal{I}$-complete if $X_i \cup X_j$ is a dependent set for each $i \neq j$;
- $\overline{\mathcal{I}}$-partition of X if X_i is a dependent set for each i;
- $\overline{\mathcal{I}}$-indivisible if X_i is an indivisible dependent set for each i.

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Partition numbers

Definition

Let $I = (X, \mathcal{I})$ be an independence system.

- The chromatic number $\chi(I)$ of I is the smallest order of an $\mathcal{I}$-partition of X;
- The achromatic number $\psi(I)$ of I is the largest order of an $\mathcal{I}$-complete $\mathcal{I}$-partition of X;
- The dual chromatic number $\overline{\chi}(I)$ of I is the largest order of an $\overline{\mathcal{I}}$-partition of X;
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Results

Theorem Cockayne, Miller, Prins

If $I = (X, \mathcal{I})$ is an independence system, then for all n satisfying $\chi(I) \leq n \leq \psi(I)$, X has an $\mathcal{I}$-complete $\mathcal{I}$-partition of order n.

Theorem

If $I = (X, \mathcal{I})$ is an independence system, then for all n satisfying $\overline{\psi}(I) \leq n \leq \overline{\chi}(I)$, X has an indivisible $\overline{\mathcal{I}}$-partition of order n.

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