Zbigniew Grande*,
Institute of Mathematics, Bydgoszcz Academy, Plac Weyssenhoffa 11,
85-072 Bydgoszcz, Poland.

ON THE POINTWISE, DISCRETE AND TRANSFINITE LIMITS OF SEQUENCES OF CLOSED GRAPH FUNCTIONS

Abstract

In this article we prove that if a function $f : X \to \mathcal{R}$ is the pointwise (discrete) [transinite] limit of a sequence of real functions f_n with closed graphs defined on complete separable metric space X then f is the pointwise (discrete) [transfinite] limit of a sequence of continuous functions. Moreover we show that each Lebesgue measurable function $f : \mathcal{R} \to \mathcal{R}$ is the discrete limit of a sequence of functions with closed graphs in the product topology $T_d \times T_e$, where T_d denotes the density topology and T_e the Euclidean topology.

We say that a function $f : X \to Y$, where X and Y are topological spaces, is a function with closed graph, if the graph of the function f, i.e. the set

$$G(f) = \{(x, y) \in X \times Y; y = f(x)\},$$

is a closed subset of the product $X \times Y$.

Let $\mathcal{R}$ be the space of all reals with the Euclidean topology T_e. In the paper [10] Kostyrko proves that every function $f : (X, T_X) \to (R, T_e)$ (shortly $f : X \to R$) defined on a normal topological space X, with a closed graph is the limit of a sequence of continuous functions $f_n : X \to R$, i.e. it is of the first class of Baire.

It is also obvious to observe that the uniform limit of a sequence of functions $f_n : X \to R$ with closed graphs, has the closed graph ([6]).

In this article I prove that on a separable complete metric space (X, ρ) the pointwise (resp. discrete) limit of a sequence of functions $f_n : X \to R$ with closed graphs is the pointwise (resp. discrete) limit of a sequence of real continuous functions on X.

Theorem 1 *Let (X, ρ) be a complete metric space. If a function $f : X \to R$ is the pointwise limit of a sequence of functions $f_n : X \to R$ with a closed graph then f is of the first Baire class.*

AMS subject classification: 26A15, 54C30. **Key words:** Function with closed graph, discrete convergence, pointwise convergence, transfinite convergence, density topology.

*Supported by Bydgoszcz Academy grant 2001

Proof. By Theorem 1 from [8] it suffices to prove that for every nonempty perfect set $A \subset X$ and for each positive real η there is an open interval I such that

$$I \cap A \neq \emptyset \text{ and } osc_{I\cap A} f < \eta.$$

Let $A \subset X$ be a nonempty perfect set and let η be a positive real. For each point $x \in A$ there is a postive integer $n(x)$ such that

$$|f_n(x) - f(x)| < \frac{\eta}{8} \text{ for } n > n(x).$$

For $n = 1, 2, \ldots$, let

$$A_n = \{x \in A; n(x) = n\}.$$

Since A with the restricted metric $\rho/(A \times A)$is a complete metric space and

$$A = \bigcup_{n=1}^{\infty} A_n,$$

there is a positive integer k such that the set A_k is of the second category in A. Consequently, the interior (in the space A) $int_A(cl(A_k))$ of the closure $cl(A_k)$ of the set A_k is nonempty and there is an open set J such that

$$\emptyset \neq cl(A_k \cap J) = cl(J \cap A).$$

Consider a function f_m with $m > k$. Since the graph $G(f_m/A)$ of the restricted function f_m/A is closed, the set $D(f_m/A)$ of all discontinuity points of the restricted function f_m/A is nowhere dense in A (see [1, 7]. So there is an open set $I \subset J$ such that

$$I \cap A \neq \emptyset \text{ and } osc_{I\cap A} f_m < \frac{\eta}{4}$$

and the restricted function f_m/A is continuous at every point of the set $I \cap A$. For each positive integer $j > k$ and for each point $x \in I \cap A_k$ the inequality $|f_j(x) - f(x)| < \frac{\eta}{8}$ is true. So, for $j > k$ and $x \in I \cap A_k$ we obtain

$$|f_j(x) - f_m(x)| \leq |f_j(x) - f(x)| + |f(x) - f_m(x)| < \frac{\eta}{8} + \frac{\eta}{8} = \frac{\eta}{4}.$$

Since the restricted function $f_m/(I \cap A)$ is continuous and the set $I \cap A_k$ is dense in $I \cap A$, the restricted functions $f_j/(I \cap A)$, $j > k$, must be also continuous. Moreover for $x, y \in I \cap A$ and for $j > k$ the inequality

$$|f_j(x) - f_j(y)| \leq |f_j(x) - f_m(x)| + |f_m(x) - f_m(y)| + |f_m(y) - f_j(y)| < \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{4} = \frac{3\eta}{4}$$

is true. We will show that $osc_{I\cap A} f \leq \frac{3\eta}{4} < \eta$. If $osc_{I\cap A} f > \frac{3\eta}{4}$ then there are points $u, v \in I \cap A$ such that $|f(u) - f(v)| > \frac{3\eta}{4}$. But

$$|f(u) - f(v)| = |\lim_{j\to\infty} f_j(u) - \lim_{j\to\infty} f_j(v)| = \lim_{j\to\infty} |f_j(u) - f_j(v)| \leq \frac{3\eta}{4},$$

and this contradiction finishes the proof.

Now we will describe the discrete convergence of sequences of functions with closed graphs.

A sequence of functions $f_n : X \to \mathcal{R}$ discretely converges to a function f ([4]) if for each point $x \in X$ there is a positive integer $n(x)$ such that $f_n(x) = f(x)$ for $n > n(x)$.

It is known ([4, 8, 9]) that a function $f : X \to \mathcal{R}$ defined on separable complet metric space X is the discrete limit of a sequence of continuous functions $f_n : X \to \mathcal{R}$ if and only if for each nonempty closed set $A \subset X$ there is a nonempty open set $G \subset X$ such that $G \cap A \neq \emptyset$ and the restricted function $f/(G \cap A)$ is continuous.

Theorem 2 *Let (X, ρ) be a separable complete metric space. If a function $f : X \to \mathcal{R}$ is the discrete limit of a sequence of functions $f_n : X \to \mathcal{R}$ with closed graphs then f is the discrete limit of a sequence of real continuous functions defined on X.*

Proof. Let $A \subset X$ be a nonempty perfect set. For each point $x \in A$ there is a positive integer $n(x)$ such that $f_n(x) = f(x)$ for $n > n(x)$. For $n = 1, 2, \ldots$ put

$$A_n = \{x \in A; n(x) = n\}.$$

Since $(A, \rho/(A \times A))$ is a complete space and

$$A = \bigcup_{n=1}^{\infty} A_n,$$

there is an integer $k > 0$ such that the set A_k is of the second category in A. So there is an open set $J \subset X$ such that

$$J \cap A \neq \emptyset \text{ and } cl(A_k \cap J) = cl(A \cap J).$$

Consider a function f_m with $m > k$. The set $D(f_m/(I \cap A))$ of discontinuity points of $f_m/(J \cap A))$ is nowhere dense in $J \cap A$, so there is an open set $I \subset J$ such that $I \cap A \neq \emptyset$ and the restricted funtion $f_m/(I \cap A)$ is continuous. Since the graphs $G(f_n/A$ are closed and for $j > k$ we have

$$f_j(x) = f_m(x) \text{ for } x \in A_k \cap I \text{ and } cl(I \cap A) = cl(I \cap A_k),$$

the restricted functions $f_j/(I \cap A)$, $j > k$, are continuous and

$$f_j/(I \cap A) = f_m/(I \cap A).$$

Consequently, the restricted function $f/(I \cap A) = f_m/(I \cap A)$ is continuous and the proof is completed.

Now we consider the transfinite convergence. Let ω_1 be the first uncountable ordinal. We will say [12]) that a transfinite sequence of functions $f_\alpha : X \to \mathcal{R}$, $\alpha < \omega_1$, converges to a function f if for each point $x \in X$ there is a countable ordinal $\beta(x)$ such that $f_\alpha(x) = f(x)$ for countable ordinals $\alpha > \beta(x)$.

In the proof of next theorem we will apply the following lemma.

Lemma 1 *Let (X, ρ) be a separable complete metric space and let $F \subset X$ be a nonempty closed set. If a transfinite sequence of functions $f_\alpha : X \to \mathcal{R}$, $\alpha < \omega_1$, with closed graphs converges to a function f then there are an open set U with $U \cap F \neq \emptyset$ and a countable ordinal β such that*

$$f_\alpha(x) = f(x) \text{ for } x \in U \cap F \text{ and } \omega_1 > \alpha > \beta.$$

Proof. There is a countable set $A \subset F$ such that $cl(A) = F$. Let $\beta < \omega_1$ be an ordinal such that

$$f_\alpha(x) = f(x) \text{ for } x \in A \text{ and } \omega_1 > \alpha > \beta.$$

Since the graphs of restricted functions f_α/F are closed in the product space $F \times \mathcal{R}$, there is an open and dense in F subset $B \subset F$ such that

$$f_\alpha(x) = f(x) \text{ for } x \in B \text{ and } \omega_1 > \alpha > \beta.$$

Let $B = U \cap F$, where U is open in X. Then the set U satisfies all requirements and the proof is completed.

Theorem 3 *Let (X, ρ) be a separable complete metric space. If a transfinite sequence of functions $f_\alpha : X \to \mathcal{R}$, $\alpha < \omega_1$, with closed graphs converges to a function f then there is a countable ordinal β such that $f_\alpha = f$ for all countable ordinals $\alpha > \beta$.*

Proof. Let $\mathcal{B}$ be a countable basis of open sets in X. By the above Lemma and the transfinite induction we find a transfinite sequence of open sets $U_\alpha \in \mathcal{B}$,

$\alpha < \alpha_0$, and a transfinite increasing sequence of countable ordinals $\beta(\alpha)$, $\alpha < \alpha_0$, such that

$$X = \bigcup_{\alpha<\alpha_0} U_\alpha,$$

$$V_\alpha = U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta \neq \emptyset$$

and

$$f_\alpha(x) = f(x) \text{ for } x \in V_\alpha \text{ and } \omega_0 \leq \alpha > \beta(\alpha).$$

Since α_0 is a countable ordinal, there is a countable ordinal $\gamma > \beta(\alpha)$ for all $\alpha < \alpha_0$. Obviously, $f_\alpha = f$ for all countable $\alpha > \gamma$ and the proof is completed.

Now we consider the pointwise and discrete convergence of sequences of functions with closed graphs in the case of the density topology.

A point $x \in \mathcal{R}$ is said an outer density point of a set $A \subset \mathcal{R}$ if

$$\lim_{h\to 0^+} \frac{\mu_e([x-h, x+h] \cap A)}{2h} = 1,$$

where μ_e denotes the Lebesgue outer measure on $\mathcal{R}$.

If a set $A \subset \mathcal{R}$ is measurable (in the Lebesgue sense) then each outer density point of A is said a density point of A.

The family T_d of all measurable sets $A \subset \mathcal{R}$ such that each point $x \in A$ is a density point of A, is a topology said the density topology ([3, 13]). The space $(\mathcal{R}, T_d)$ is completely regular but it is not normal ([13]).

Now we will consider functions $f : (\mathcal{R}, T_d) \to (\mathcal{R}, T_e)$.

Theorem 4 *If the graph $G(f)$ of a function $f : \mathcal{R} \to \mathcal{R}$ is closed in the product topology $T_d \times T_e$ then f is measurable.*

Proof. By Davies lemma from [5] it suffices to show that for each measurable set $A \subset \mathcal{R}$ of positive measure and for each positive real η there is a measurable set $B \subset A$ of positive measure such that $osc_B f \leq \eta$.

Suppose, to the contrary, that there are a real $\eta > 0$ and a measurable set $A \subset \mathcal{R}$ such that $\mu(A) > 0$ and $osc_B f > \eta$ for every measurable subset $B \subset A$ of positive Lebesgue measure $\mu(B)$.

There is a closed interval $[c, d]$ such that

$$d - c < \frac{\eta}{2} \text{ and } \mu_e(f^{-1}([c, d])) > 0.$$

Let $H \in T_d$ be a nonempty set such that every measurable set $B \subset H \setminus f^{-1}([c,d])$ is of measure zero. There is a point $x \in H$ with $f(x) \in \mathcal{R} \setminus [c,d]$. Let

$$y = \sup\{\inf_B f; B \subset f^{-1}([c,d]) \text{ and } x \text{ is an outer density point of } B\}.$$

Obviously

$$y \in [c,d] \text{ and } (x,y) \in \mathcal{R}^2 \setminus G(f).$$

We will show that $(x,y) \in cl(G(f))$ with respect to $T_d \times T_e$. For this let sets $U \in T_d$ and an open interval V be such that $x \in U$ and $y \in V$. From the definition of y follows that there is a set $B \subset f^{-1}([c,d])$ such that x is an outer density point of B and $\inf_B f \in V$. Then x is also an outer density point of the set $B \cap U$ and

$$\inf_B f \leq \inf_{B \cap U} f \leq y.$$

Consequently, $\inf_{B \cap U} f \in V$ and there is a point $u \in U \cap B$ with $f(u) \in V$.

So, $(x,y) \in cl(G(f))$ relative to the topology $T_d \times T_e$ and the graph $G(f)$ is not closed. This contradiction finishes the proof.

Since measurable functions are almost everywhere approximately continuous and the sets of measure zero are nowhere dense and closed in the density topology T_d, we obtain:

Corollary 1 *If the graph of a function $f : \mathcal{R} \to \mathcal{R}$ is closed in the product topology $T_d \times T_e$ then the set $D(f)$ of all T_d-discontinuity points of f is closed and nowhere dense in T_d.*

Functions $f : \mathcal{R} \to \mathcal{R}$ with closed graph in the topology $T_d \times T_e$ may be nonborelien.

Example.

Let C be the ternary Cantor set and let $(I_n)_n$ be an enumeration of all components of the complement $\mathcal{R} \setminus C$ such that $I_n \cap I_m$ for $m \neq n$. Let $B \subset C$ be a nonborelien set. For $n = 1, 2, \ldots$ let $f_n : I_n \to [n, \infty)$ be a continuous function such that if x is an endpoint of I_n then $\lim_{I_n \ni t \to x} f(t) = \infty$. Then the graph of the function

$$f(x) = \begin{cases} f_n(x) & for \quad x \in I_n,\ n \geq 1 \\ 1 & for \quad x \in B \\ 0 & for \quad x \in C \setminus B \end{cases}$$

is closed in the product topology $T_d \times T_e$, but f is not borelien.

Theorem 5 *If $f : \mathcal{R} \to \mathcal{R}$ is measurable then there is a sequence of functions $g_n : \mathcal{R} \to \mathcal{R}$ with closed graphs in the topology $T_d \times T_e$ which discretely converges to f.*

Proof. By Lusin Theorem there are closed (in T_e) sets A_n, $n \geq 1$, such that the restricted functions f_n/A_n are T_e-continuous,

$$A_n \subset A_{n+1} \text{ for } n = 1, 2, \dots \text{ and } \mu_e(\mathcal{R} \setminus \bigcup_{n=1}^{\infty} A_n) = 0.$$

The set

$$A = \mathcal{R} \setminus \bigcup_{n=1}^{\infty} A_n$$

is an G_δ-set of measure zero. So for each integer $n \geq 1$ there is an G_δ-set $E_n \supset A$ of measure zero which contains all endpoints of components of the complement $\mathcal{R} \setminus A_n$. By Zahorski Lemma ([3]) for $n \geq 1$ there are approximately continuous functions $f_n : \mathcal{R} \to [0, 1]$ (i.e. f_n are continuous as applications from $(\mathcal{R}, T_d)$ to $(\mathcal{R}, T_e)$) such that $f_n(x) = 0$ for $x \in E_n$, $f_n(x) > 0$ otherwise on $\mathcal{R}$ and f_n are T_e-continuous at points $x \in E_n$.

Let $(I_{k,n})_k$ be an enumeration of all components of the complement $\mathcal{R} \setminus A_n$ such that $I_k \cap I_j = \emptyset$ for $k \neq j$. For $n \geq 1$ define

$$g_n(x) = \begin{cases} f(x) & for \quad x \in A_n \cup E_n \\ \max(k, \frac{1}{f_n(x)}) & for \quad x \in I_{k,n} \setminus E_n, \ k \geq 1. \end{cases}$$

Then the graphs $G(g_n)$ are closed in $T_d \times T_e$ for $n \geq 1$ and the sequence $(g_n)_n$ discretely converges to f.

Remark 1 *Since the pointwise limit f of a sequence of approximately continuous functions $f_n : \mathcal{R} \to \mathcal{R}$ is of the second Baire class and since there are nonborelien measurable functions, Theorems 1 and 2 are not true for the case the topology $T_d \times T_e$.*

Theorem 6 *Assume that Continuum Hypothesis (HC) is true. For each function $f : \mathcal{R} \to \mathcal{R}$ there are functions $f_\alpha : \mathcal{R} \to \mathcal{R}$ with closed graphs in the topology $T_d \times T_e$, where $\alpha < \omega_1$, such that the transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ converges to a function f.*

Proof. Enumerate all reals in a transfinite sequence $(a_\alpha)_{\alpha<\omega_1}$ such that $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$.

For $\alpha < \omega_1$ let

$$A_\alpha = \{a_\beta; \beta < \alpha\}$$

and let $g_\alpha : \mathcal{R} \to [0,1]$ be an approximately continuous function such that

$$\mu_e(g_\alpha^{-1}(0)) = 0 \text{ and } A_\alpha \subset g_\alpha^{-1}(0),$$

and g_α is continuous at each point x at which $g_\alpha(x) = 0$. Then the function

$$f_\alpha(x) = \begin{cases} \frac{1}{g_\alpha(x)} & if \quad g_\alpha(x) \neq 0 \\ f(x) & otherwise\ on \quad \mathcal{R} \end{cases}$$

has the closed graph $G(f_\alpha)$ in the topology $T_d \times T_e$ and

$$f_\alpha(x) = f(x) \text{ for } x \in A_\alpha.$$

Ewidently, the transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ converges to f. This completes the proof.

In connection with last theorem remember ([11]) that a function $f : \mathcal{R} \to \mathcal{R}$ is the limit of a transfinite sequence of approximately continuous functions $f_\alpha : \mathcal{R} \to \mathcal{R}$ if and only if it is of the first Baire class.

References

[1] Baggs I.; *Functions with a closed graphs*, Proc. Amer. Math. Soc. 43 (1974), 439–442.

[2] Borsik J., Dobos J. and Repicky M.; *Sums of quasicontinuous with closed graphs*, Real Analysis Exchange 25 No. 2 (1999-2000), 679–690.

[3] Bruckner A.M.; *Differentiation of real functions*, Lectures Notes in Math. 659, Springer-Verlag, Berlin 1978.

[4] Davies R. O.; *Approximate continuity implies measurability*, Math. Proc. Camb. Philos. Soc. 73 (1973), 461–465.

[5] A. Császár and M. Laczkovich; *Discrete and equal convergence*, Studia Sci. Math. Hungar. 10 (1975), 463–472.

[6] Dobos J.; *A note on the functions the graph of which are closed sets*, Acta Math. Univ. Comenian. 40-41 (1982), 285–288.

[7] Dobos J.; *On the set of points of discontinuity for functions with closed graphs*, Cas. pest. mat. 110 (1985), 60–68.

[8] Grande Z.; *Baire functions and their restrictions to special sets*, Math. Slovaca 43 (1993), 447–453.

[9] Kirchheim B.; *Baire one star functions*, Real Analysis Exchange 8 (1992//93), 385–389.

[10] Kostyrko P.; *A note on the functions with closed graphs*, Cas. pest. mat. 94 (1969), pp. 202–205.

[11] Lipiński J. S.; *On transfinite sequences of mappings*, Cas. pest. mat. 101 (1976), 153–158.

[12] Sierpiński W.; *Sur les suites transfinies convergentes de fonctions de Baire*, Fund. Math. 1 (1920), 132–141.

[13] Tall F. D.; *The density topology*, Pacific J. Math. 62 (1976), pp.275-284.