Centralized broadcasting in radio networks with k-degenerate reachability graphs

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Abstract. *We consider deterministic radio broadcasting in radio networks whose nodes have full topological information about network and the reachability graph of a network is k-degenerate. The goal is to design a polynomial algorithm which produces a fast radio broadcast schedule with respect to a reachability graph G and a source $s \in V(G)$. The length of produced schedule is considered as the measure of efficiency. For each k, $k \geq 2$, we show that there are k-degenerate graphs with n nodes for which every radio broadcast schedule has the length $\Omega(\log n)$. Finally, we design an algorithm producing a radio broadcast schedule of the length $O_k(D \log n/D)$ for each k-degenerate graph with n nodes and the eccentricity D of a source.*

1 Introduction

A *radio network* is a collection of autonomous stations that are referred as *nodes*. The nodes communicate via sending messages. Each node is able to receive and transmit messages, but it can transmit messages only to nodes, which are located within its transmission range. The network can be modeled by a directed graph called *reachability graph* $G = (V, E)$. The vertex set of G consists of the nodes of the network and two vertices $u, v \in V$ are connected by an edge $e = (u, v)$ if and only if the transmission of the node u can reach the node v. In such a case the node u is called a *neighbour* of the node v. If the transmission power of all nodes is the same, then the reachability graph is symmetric, i.e. a symmetric radio network can be modeled by an undirected graph.

Nodes of a network work in synchronised steps (time slots) called *rounds*. In every round, a node can act either as a *receiver* or as a *transmitter*. A node u acting as transmitter sends a message, which can be potentially received by each its neighbour. In the given round, a node, acting as a receiver, receives a message only if it has exactly one transmitting neighbour. The received message is the same as the message transmitted by the transmitting neighbour. If in the given round, a node u has at least two transmitting neighbours we say that a *collision* occurs at node u. In the case, when the nodes can distinguish collision from silence, we say that they have an *availability of collision detection*. It is also assumed that a node can determine its behavior in the following round within the actual round.

The goal of *broadcasting* is to distribute a message from one distinguished node, called a *source*, to all other nodes. Remote nodes of the network are informed via intermediate nodes. A *radio broadcast schedule* for a given network prescribes in which step which nodes transmit. Its length corresponds to the time required to complete the broadcast operation, i.e. to inform all nodes of the network. The time, required to complete an operation, is important and widely studied parameter of mostly every communication task. In this paper we consider the length of a broadcast schedule as a function of two parameters of radio network: number of nodes (denoted as n), and the largest distance from the source to any other node of the network (denoted as D).

According to different features of the stations forming a radio network, many models of radio networks have been developed and studied. They differ in used communications scenarios and initial knowledge assumed for nodes. The overview of the models of radio networks can be found e.g. in [13]. In this paper we focus on the deterministic broadcasting in radio networks whose nodes have full topological knowledge about the network. Each node has as an initial knowledge: its unique integer identifier, an identifier of the source node and a labeled copy of underlying reachability graph. Using the same algorithm to produce a broadcast schedule in each node with the same inputs (identifier of the source and underlying reachability graph), the obtained broadcast schedule can be considered as a broadcasting controlled from one central. Thus broadcasting can be performed by nodes according to produced broadcast schedule in a distributed

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way. The mentioned setting is refereed as *centralized radio broadcasting* or *broadcasting in known topology radio networks*. In this setting, the goal is to design a deterministic polynomial algorithm which produces a fast broadcast schedule for a given input.

In this paper we consider only the radio networks, whose underlying reachability graph has a special topology, namely it is k-degenerate for a fixed positive integer k.

1.1 Related work

The study of communication in known topology radio network was initiated in the context of broadcasting problem. In [8] the authors presented a deterministic polynomial algorithm producing a broadcast schedule of the length $O(D \log^2 n)$, for any graph with n nodes and diameter D. In [1] the lower bound $\Omega(\log^2 n)$ of the length of broadcasting schedule was proved for a family of graphs with diameter 2. Later in [5] the authors proposed a method improving the time of broadcasting in the case when reachability graph is undirected. The method is based on partitioning of the underlying reachability graph into clusters with smaller diameter and applying broadcast schedules produced by known algorithms in each cluster separately. This method was later improved in [4]. Applying the deterministic algorithm from [10], which produces a broadcast schedule of the length $O(D \log n + \log^2 n)$, method from [4] computes a broadcast schedule of the length $O(D + \log^4 n)$. Very recently these results have been further improved for undirected graphs in [7] to $O(D + \log^3 n)$. Finally in [11], the authors proposed an algorithm producing a radio broadcast schedule of the asymptotically optimal length $O(D + \log^2 n)$.

In the case when underlying reachability graph is undirected and planar, centralized broadcasting was firstly investigated in [4] where an algorithm producing schedules of the length $O(D+\log^4 n)$ was proposed. Recently, independently in [7] and [6], the algorithms producing a schedule of the length $3D$ were designed. It is remarked in [6] that a broadcast schedule of the length $3D$ can be produced also in the case of directed and planar reachability graph. This sharp gap between the time of broadcasting in general case and in the case of planar graphs was our main motivation to study centralized radio broadcasting in networks whose underlying reachability graph is k-degenerate (note that every planar graph is 5-degenerate).

In this paper we use also the notion of selective families, which has been introduced in [9]. More precisely, we use ad-hoc selective families whose computing has been studied in [3].

1.2 Terminology and preliminaries

Firstly we recapitulate some concepts of graph theory. We assume the standard graph terminology. Next we formulate the broadcasting problem more formally and we show its relationship to selective families. Finally we describe a class of k-degenerate graphs.

The set $N(u) = \{v \in V(G) : (v, u) \in E(G)\}$ we shall denote as *neighbourhood* of a node u. The *distance* of two nodes u, v (denoted by $dist(u, v)$) is the length of a shortest $u - v$-path in the underlying reachability graph. The *eccentricity* of a node v is defined as $ecc(v) = \max\{dist(v, u) : u \in V(G)\}$. We briefly denote the eccentricity of the source node s by $D = ecc(s)$. It is not difficult to see that all nodes of a reachability graph G can be partitioned into *layers* with respect to their distances from the source s. Hence, we can define the sets

$$L_i = \{v \in V(G) : dist(s, v) = i\},\ i = 0, 1, \ldots, ecc(s).$$

Definition 1. *Let $G = (V, E)$ be a directed graph and $R \subseteq V$ be a subset of nodes. The set of nodes informed by R, denoted by $I(R)$, is the set*

$$I(R) = \{v \in V : \text{there exists the unique } x \in R \text{ such that } v \in N(x)\}.$$

For a singleton set $R = \{x\}$, $I(R) = I(\{x\}) = N(x)$.

Definition 2. *Let $G = (V, E)$ be a directed graph. A sequence of sets $\Pi = (R_1, \ldots, R_q)$ is called a radio broadcast schedule with respect to the reachability graph G and a source $s \in V$ if and only if the following holds:*

1. *$R_i \subseteq V$, for every $i = 1, 2, \ldots, q$;*
2. *$R_1 = \{s\}$;*
3. *$R_{i+1} \subseteq \bigcup_{j=1}^{i} I(R_j)$, for every $i = 1, 2, \ldots, q - 1$;*
4. *$V = \bigcup_{j=1}^{q} I(R_j)$.*

The length of the schedule Π is $q = |\Pi|$.

The property 2 of the previous definition guaranties that in the first round only the source transmits a message. From the property 3 it follows that only informed nodes can transmit. Finally, the property 4 implies that all nodes become informed after performing the broadcast schedule, i.e. every node receives a message in at least one round of the schedule. Note that it is supposed that in the graph G there is a path from the source s to any other node of the network.

For a given collection $\mathcal{F}$ of subsets of $[n] = \{1, \ldots, n\}$, a *selective family for $\mathcal{F}$* is a collection $\mathcal{S}$ of subsets of $[n]$ such that for any $F \in \mathcal{F}$ there exists $S \in \mathcal{S}$ such that $|F \cap S| = 1$. The relationship between selective families and broadcasting in radio networks can be expressed in the following way: suppose that the labels of nodes are pairwise distinct integers from the set $[n]$, where n is the number of nodes. Let I be a set of informed nodes. Assume that there is a subset U of uninformed nodes such that each node $u \in U$ has at least one informed neighbour. Note that if there is at least one uninformed node then such set $U \neq \emptyset$ always exists. We can construct a collection $\mathcal{F}_U = \{N(u) \cap I : u \in U\}$, i.e. for each node the set of labels of its informed neighbours is a member of the collection $\mathcal{F}_{\mathcal{U}}$. Let $\mathcal{S}_U = \{S_1, \ldots, S_m\}$ be a selective family for $\mathcal{F}_U$. Clearly, a schedule of the length m, such that in the round i exactly the nodes with the labels in the set S_i transmit, ensures that all nodes belonging to the set U become informed. Intuitively, the smaller selective family for a given collection $\mathcal{F}$ we are able to compute, the shorter radio broadcast schedule we are able to produce.

In this paper we shall use the result from [3]:

Theorem 1. *[3] There exists an algorithm that for a given collection $\mathcal{F}$ of subsets of $[n]$, each of size in the range $[\Delta_{min}, \Delta_{max}]$, computes a selective family $\mathcal{S}$ for $\mathcal{F}$ of size $O((1+\log(\Delta_{max}/\Delta_{min})).\log|\mathcal{F}|)$. The time complexity of the algorithm is*

$$O(n^2|\mathcal{F}|\log|\mathcal{F}|.(1+\log(\Delta_{max}/\Delta_{min}))).$$

Now we define and describe a class of k-degenerate graphs.

Definition 3. *Let k be a non-negative integer. A graph G is called k-degenerate (we write $G \in \mathcal{D}_k$), if for each subgraph H of G, the minimum degree of H does not exceed k.*

The following value plays the fundamental role in the theory of k-degenerate graphs:

$$s(G) = \max_{H \subseteq G} \min_{v \in V(H)} \deg_H(v).$$

This number is called *Szekeres-Wilf number* and it is easy to see that G is k-degenerate if and only if $s(G) \leq k$. The definition implies that each subgraph of k-degenerate graph is k-degenerate as well (for more details see [2]) and moreover for each graph G there is a number k such that G is k-degenerate.

Proposition 1. *[12] A graph G of order $k+m$ is k-degenerate if and only if the vertex set $V(G)$ can be labeled $v_1, v_2, \ldots, v_{k+m}$ such that in the subgraph $\langle\{v_i, v_{i+1}, \ldots, v_{k+m}\}\rangle$ of G $\deg(v_i) \leq k$ for each $i = 1, 2, \ldots, m-1$.*

Note that the labeling of k-degenerate graph G satisfying the previous proposition can be computed in such a way, that in every step we take out one node of the lowest degree. Obviously this computation takes polynomial time. Also note that k-degenerate graphs have no general bound on the maximal degree of a node. On the other side it was shown in [12], that the number of edges of a k-degenerate graph is at most $kn - \binom{k+1}{2}$ where n is the number of nodes.

2 Lower bound

In this section we show a lower bound concerning the time of broadcasting in radio networks, whose reachability graph is k-degenerate for $k \geq 2$. In particular, we show that there is a subclass of 2-degenerate graphs, such that for each graph of this subclass every radio broadcast schedule has the length $\Omega(\log n)$.

At first we define a set of graphs $\mathcal{G} = \{G_m : m \geq 2\}$. For a fixed integer m, $m \geq 2$, the graph G_m is constructed from the graph K_m with vertex set $V(K_m) = \{v_1, \ldots, v_m\}$ (the complete graph on m vertices) as follows: we add a new node s to K_m and we join it to every node of K_m. Next we subdivide every edge $e_{i,j} = (v_i, v_j) \in E(K_m)$ by a new node $u_{i,j}$. Formally, $G_m = (V_m, E_m)$ is an undirected graph with the vertex set $V_m = \{s, v_1, \ldots, v_m\} \cup \{u_{i,j} : 1 \leq i < j \leq m\}$ and the edge set $E_m = \{(s, v_i) : 1 \leq i \leq m\} \cup \{(v_i, u_{i,j}), (v_j, u_{i,j}) : 1 \leq i < j \leq m\}$.

With respect to the source node s, the graph G_m can be partitioned into layers $L_0 = \{s\}$, $L_1 = \{v_i : 1 \leq i \leq m\}$ and $L_2 = \{u_{i,j} : 1 \leq i < j \leq m\}$. Each layer forms an independent set. Obviously, the radius of G_m is 2. Since every node, except the source s, has degree at most 2, the graph G_m is a 2-degenerate graph with $(m^2 + m + 2)/2$ nodes.

In the following lemma we show that it is not possible to complete radio broadcasting in the graph G_m with the source s in less than $\lfloor \log n \rfloor + 1$ rounds.

Lemma 1. *Any radio broadcast schedule for graph G_m with respect to the source $s \in V(G_m)$ has the length at least $\lfloor \log m \rfloor + 1$.*

Proof. We fix a radio broadcast schedule $\Pi = (R_1, \ldots, R_q)$. Since $R_1 = \{s\}$, only the source s transmits in the first round. This transmission informs all nodes belonging to the layer L_1. Hence the rest of the schedule informs only the nodes of the layer $L_2 = \{u_{i,j} : 1 \leq i < j \leq m\}$ by the transmissions of nodes of the layer L_1. According to the schedule Π we can associate a binary sequence $s_i = (s_i^1, \ldots, s_i^{q-1})$ of length $q-1$ with each node $v_i \in L_1$. We set s_i^r to 1 if and only if $v_i \in R_{r+1}$. Otherwise we set s_i^r to 0. It is easy to see that a node $u_{i,j} \in L_2$ receives a message exactly in each round r such that $s_i^{r-1} \neq s_j^{r-1}$. Since Π is a radio broadcast schedule, every node $u_{i,j} \in L_2$ is informed and it receives a message in at least one round.

Thus for each i, j, $i \neq j$, the binary sequences s_i and s_j should differ in at least one position, i.e. $s_i \neq s_j$. It implies that there are exactly $m = |L_1|$ different sequences associated with nodes of the layer L_1.

Clearly, we can construct at most 2^{q-1} different binary sequences of the length $q-1$. Suppose now that $q - 1 < \lfloor \log m \rfloor$. It implies that $2^{q-1} < 2^{\lfloor \log m \rfloor} \leq 2^{\log m} = m$, i.e. $2^{q-1} < m$. The inequality contradicts the fact that we have m different binary sequences of the length $q-1$. □

Theorem 2. *There is a subclass $\mathcal{C}$ of 2-degenerate graphs with radius 2 such that:*

1. *for every integer n, $n \geq 9$, there is a graph $G \in \mathcal{C}$ such that $|V(G)| = n$;*
2. *for every graph $G \in \mathcal{C}$ there is a node $s \in V(G)$ such that every radio broadcast schedule with respect to the graph G and the source s has the length $\Omega(\log n)$, where $n = |V(G)|$ is the number of nodes.*

Proof. Fix an arbitrary real number $c \in (0, \frac{1}{4})$. For each $n \geq 9$, we show that there are 2-degenerate graphs on n nodes with the radius 2, for which every radio broadcast schedule has the length at least $\lfloor c \log n \rfloor$ rounds. It is easy to see that the chosen c, n and $m = \lceil n^c \rceil$ satisfy the inequality $n - (m^2 + m + 2)/2 \geq 0$.

Let $G = (V, E)$ be a graph with n nodes constructed from the graph G_m, where $m = \lceil n^c \rceil$, by adding $n - (m^2 + m + 2)/2$ new nodes and joining them to the node $s \in V(G_m)$. Then G has the vertex set $V(G) = V(G_m) \cup \{w_1, \ldots, w_{n-(m^2+m+2)/2}\}$ and the edge set $E(G) = E(G_m) \cup \{(s, w_i) : 1 \leq i \leq n - (m^2 + m + 2)/2\}$ and obviously graph G is 2-degenerate graph with radius 2. Let $s \in V(G)$ be the source. Since there are no edges between $V(G_m) \setminus \{s\}$ and $(V(G) \setminus V(G_m)) \setminus \{s\}$, the broadcast operation is performed in the subgraph G_m separately. Previous lemma implies that it is not possible to complete broadcasting in G_m (and also in G) in less than $\lfloor \log m \rfloor + 1 \geq \log n^c \geq \lfloor c \log n \rfloor$ rounds. □

Note that, in the previous proof there are more ways how to construct a graph G satisfying desired properties. In more general construction we add $n - (m^2 + m + 2)/2$ new nodes to the graph G_m. Next we add new edges to the graph G between the nodes of the set $W = (V(G) \setminus V(G_m)) \cup \{s\}$ (i.e. between newly created nodes and the source s) in such a way that the induced graph $H = \langle W \rangle_G$ is connected 2-degenerate graph satisfying $ecc_H(s) \leq 2$.

Since $\mathcal{D}_2 \subset \mathcal{D}_k \subset \mathcal{D}_{k+1}$ for each $k > 2$ (see [12]), the following holds:

Corollary 1. *Let k be a positive integer, $k \geq 2$. There is a subclass $\mathcal{C}$ of k-degenerate graphs with radius 2 such that:*

1. *for every integer n, $n \geq 9$, there is a graph $G \in \mathcal{C}$ such that $|V(G)| = n$;*
2. *for every graph $G \in \mathcal{C}$ there is a node $s \in V(G)$ such that every radio broadcast schedule for G with respect to the source s has the length $\Omega(\log n)$, where $n = |V(G)|$ is the number of nodes.*

3 Upper bound

In this section we focus on the upper bound of the length of a radio broadcast schedule. We present algorithms producing radio broadcast schedules for k-degenerate input reachability graph.

Consider a class of 1-degenerate graphs (remark that every connected 1-degenerate graph is a tree). In such a case we can construct a trivial radio broadcast schedule $\Pi = (R_1, \ldots, R_q)$ with respect to a graph (tree) G and a source $s \in V(G)$ as follows:

1. $R_1 := \{s\}$;
2. $R_{i+1} := I(R_i) \setminus \bigcup_{j=1}^{i-1} I(R_j)$, for $i \geq 1$.

The condition 2 yields that in the round $i+1$ a message is transmitted by all nodes which receive a message in the round i for the first time. It is easy to see that there is a round p such that $R_p = \emptyset$. Letting $q := p - 1$ one can prove that Π is a radio broadcast schedule of the optimal length.

In what follows we present algorithms producing a radio broadcast schedule for graphs which belong to $\mathcal{D}_k$ for a fixed integer k, $k \geq 2$.

Theorem 3. *Let $G = (A \cup B, E) \in \mathcal{D}_k$ be a bipartite k-degenerate graph ($k \geq 2$) such that $deg_G(v) \geq 1$ for all $v \in B$. Suppose that all nodes of the partition A are informed. There is a polynomial algorithm producing a schedule of the length at most $\lceil k^2/2 \rceil + k + O((1 + \log k) \log |B|)$ such that*

- *only the nodes of A transmit*
- *it ensures that all nodes of B become informed, i.e. every node of the partition B receives a message in at least one round*

Proof. The algorithm works in two phases. During each phase a part of the resulting schedule is produced. The goal of each part is to inform all nodes in the specific subset of B.

Phase 1: Let G be an input graph and denote $n = |V(G)|$. Since G is a k-degenerate graph, according to Proposition 1, in the polynomial time we can compute labeling $v_1, v_2, \ldots, v_n$ of the nodes of G such that for each $i = 1, 2, \ldots, n$ in the induced subgraph $G_i = \langle \{v_i, v_{i+1}, \ldots, v_n\} \rangle_G$ it holds $\deg_{G_i}(v_i) \leq k$. It means that the nodes of G can be ordered in such a way that there are at most k edges from the node v_i to the nodes of set $\{v_{i+1}, \ldots, v_n\}$.

For each i, $1 \le i \le n$, we define a set:

$$N_{deg}(v_i) = \{v_j \in V(G) : (v_i, v_j) \in E(G) \wedge j > i\}$$

Note that $N_{deg}(v_i) \subseteq N(v_i)$ and $|N_{deg}(v_i)| \le k$ for each $v_i \in V(G)$. The goal of this computation phase is to produce a schedule, which ensures that each node $v_i \in B$, such that $|N(v_i) \setminus N_{deg}(v_i)| \ge 1$, becomes informed. During the computation every node $v_i \in V(G)$ has assigned one round, denoted as $round(v_i)$, such that $round(v_i) \in R \cup \{NIL\}$ where $R = \{1, \ldots, \lceil k^2/2 \rceil + k\}$. Symbol NIL is used for still undefined round. For a node $v_i \in A$, the value $round(v_i)$ denotes a round in which the node v_i will transmit a message during the first part of schedule. For a node $v_i \in B$, it denotes a round (from the other admissible) in which the node v_i receives a message. Initially, we set $round(v_i) := NIL$ for all $v_i \in V(G)$. For each node $v_m \in B$ we shall maintain the following set during the computation:

$$Receive(v_m) = \{r : \text{there exists the unique}\, v_j \in N(v_m) \text{ such that } round(v_j) = r \neq NIL\}$$

The nodes are processed in the sequential order from v_n to v_1. After a node v_i is processed the following two invariants hold:

1. for each $v_j \in A$ such that $j \ge i$ we have:
   - $round(v_j) \neq NIL$
   - $round(v_m) \neq NIL$, for all $v_m \in N_{deg}(v_j)$.
2. for each $v_j \in B$ such that $j \ge i$ it holds:
   - $Receive(v_j) = \emptyset \Rightarrow round(v_j) = NIL$
   - $Receive(v_j) \neq \emptyset \Rightarrow round(v_j) \in Receive(v_j)$

Now we show, how a node $v_i \in V(G)$ is processed. We fix $v_i \in V(G)$ and suppose that all nodes in the set $\{v_{i+1}, \ldots, v_n\}$ have been already processed.

In case that $v_i \in B$ we process the node v_i as follows. If $Receive(v_i) \neq \emptyset$ then we set $round(v_i)$ to an arbitrary element of the set $Receive(v_i)$ Otherwise, $round(v_i)$ is unchanged, i.e. $round(v_i) = NIL$.

In case that $v_i \in A$ the processing of the node is more complex. For each $v_m \in B$ we compute the set:

$$Used(v_m) = \{round(v) : v \in N_{deg}(v_m)\}$$

Next we compute the following sets:

$$Unassigned = \{v_j \in N_{deg}(v_i) : round(v_j) = NIL\}$$

$$Assigned = N_{deg}(v_i) \setminus Unassigned$$

$$Used^* = \bigcup_{v_j \in Unassigned} Used(v_j)$$

$$Used = \{round(v_j) : v_j \in Assigned\} \cup Used^*$$

Finally, we set the value $round(v_i)$ to an arbitrary element of the set $R \setminus Used$. Afterwards we set the value $round(v_j) := round(v_i)$, for all $v_j \in Unassigned$.

The first part of the schedule, which corresponds to this computation phase, is produced as follows: a node $v_i \in A$ transmits a message exactly in the round $round(v_i)$, i.e. $R_j = \{v_i \in A : round(v_i) = j\}$ for each $j \in R$.

Correctness of phase 1: We use computational induction to show that during the first phase of the algorithm both mentioned invariants hold after processing of a node.

It is not difficult to see that the claim is true after processing of a node $v_i \in B$. In the case when $v_i \in A$, we show firstly the correctness of the assignment, i.e. that $R \setminus Used \neq \emptyset$ and we are able to choose a round-number.

Let $v_m \in Unassigned$. From the definition of the set $Unassigned$ it follows that $v_m \in N_{deg}(v_i) \subseteq B$, $m > i$ and $round(v_m) = NIL$. Since $m > i$ the second invariant implies that $Receive(v_m) = \emptyset$. Thus each round-number in the set $Used(v_m)$ is assigned to at least two nodes from the set $N_{deg}(v_m)$. Otherwise, there is a node $u \in N_{deg}(v_m)$ such the value $round(u) \neq NIL$ and moreover there is no node $w \in N_{deg}(v_m)$ such that $round(u) = round(w)$. But it contradicts to $Receive(v_m) = \emptyset$. Since $|N_{deg}(v_m)| \le k$ and each round-number is used at least twice, it holds that $|Used(v_m)| \le k/2$. Again using $|N_{deg}(v_i)| \le k$ we obtain:

$$|Used| \le |Assigned| + \frac{k}{2}|Unassigned| \le \frac{k^2}{2} + k - 1.$$

Hence there is at least one free round-number and $R - Used \neq \emptyset$, i.e. the defined assignment is correct.

One can verify that according to the assignments which are created at the moment when the round-number $round(v_i)$ is determined, the first invariant holds.

Now we shall analyse the second invariant. Since during the processing of the node $v_i \in A$ we change only one round-number in the set A, it is sufficient to consider validity of conditions of the second invariant only for nodes of the set $N_{deg}(v_i) \subseteq B$. Since for each node $v_m \in Assigned$ it holds that $round(v_m) \notin R \setminus Used$, validity of the conditions remains unchanged for the nodes of the set $Assigned$. Consider a node $v_m \in Unassigned$. Since after processing of v_i it holds that $NIL \neq round(v_i) = round(v_m) \notin Used$ and $round(v_m) \notin Used \Rightarrow round(v_m) \notin Used(v_m) \subseteq Used$. Since $v_m \in Unassigned$, the first invariant implies that before processing of v_i there is no node $v_j \in N(v_m) \setminus N(v_m) \subseteq A$ such that $round(v_j) \neq NIL$. Both this facts we can summarise into the claim: There is no $v_j \in N_{deg}(v_m)$ such that $round(v_j) = round(v_m)$ and

$round(v_j) = NIL$, for all $v_j \in (N(v_m) \setminus N_{deg}(v_m)) \setminus \{v_i\}$. This claim implies that $Receive(v_m) \neq \emptyset$ and $round(v_m) \in Receive(v_m)$.

Since after processing of all nodes both invariants hold, we show that each node $v_m \in B$, which satisfies $N(v_m) \setminus N_{deg}(v_m) \neq \emptyset$, become informed after execution of the first part of schedule. Let $v_m \in B$ be a node such that $N(v_m) \setminus N_{deg}(v_m) \neq \emptyset$ and let $v_i \in A$ be a node such that $v_i \in N(v_m) \setminus N_{deg}(v_m)$. The definition of $N_{deg}(v_m)$ implies that $i < m$ and thus $v_m \in N_{deg}(v_i)$. Finally, the first invariant guaranties that $round(v_m) \neq NIL$. Hence the second invariant implies that $round(v_m) \in Receive(v_m)$, i.e. v_m receives a message in at least the round $round(v_m)$.

It follows that only the nodes $v_i \in B$, for which $N(v_i) = N_{deg}(v_i)$, can be uninformed. For such uniformed nodes it holds $|N(v_i)| = |N_{deg}(v_i)| \leq k$.

Phase 2: The goal of this phase is to inform all remaining uninformed nodes. Since for every uninformed node $v_i \in B$ it holds that $|N(v_i)| \leq k$, we can use the algorithm from Theorem 1 to produce the second part of the schedule. Using the input collection $\mathcal{F} = \{\{j : v_j \in N(v_i)\} : v_i \in B \text{ is uninformed}\}$, the algorithm produces a collection $\mathcal{S} = \{S_1, \ldots, S_p\}$ as an output, where $p = O((1 + \log k) \log |B|)$. The second part of schedule is constructed using the following definition $R_{j+|R|} = \{v_i \in A : i \in S_j\}$ for each j, $1 \leq j \leq p$, where R is the set of the size $\lceil k^2/2 \rceil + k$ which has been defined in the Phase 1. Correctness of the produced schedule follows from Theorem 1.

Complexity: The total length of produced schedule is $\lceil k^2/2 \rceil + k + O((1 + \log k) \log |B|)$ rounds. Since both phases take polynomial time, the designed algorithm is polynomial as well. □

Theorem 4. *Let $G = (V, E)$ be a directed connected k-degenerate graph ($k \geq 2$), i.e. $G \in \mathcal{D}_k$. Then there exists a polynomial algorithm producing a radio broadcast schedule of the length $D.(\lceil k^2/2 \rceil + k + O((1 + \log k) \log \frac{n}{D}))$.*

Proof. Since each subgraph of a k-degenerate graph is k-degenerate too, we use the algorithm from the proof of Theorem 3 to inform the nodes of every consecutive layer, i.e. broadcasting is scheduled layer by layer. The length of the schedule follows from the fact that $\sum_{i=1}^{D} \log |L_i|$ obtains maximal value for $|L_i| = n/D$. □

Since k is fixed constant, the previous theorem implies the following corollary:

Corollary 2. *Let $k \geq 2$ be an integer and D_k be a class of k-degenerate graphs. The there is a polynomial deterministic algorithm which produces a radio broadcast schedule of length $O_k(D \log n/D)$ with respect to a reachability graph $G \in \mathcal{D}_k$ and a source $s \in V(G)$.*

It is not difficult to see that for k-degenerate graphs with diameter $o(\log n)$ proposed algorithm produces radio broadcast schedules of shorter length than known algorithms for general case.

4 Conclusion

For a fixed positive integer k, we focused on deterministic centralized radio broadcasting in radio networks whose reachability graphs are k-degenerate. In view of the presented lower bound, the proposed algorithm produces asymptotically optimal radio broadcast schedules for k-degenerate graphs of constant diameter. The lower bound shows that planar graphs differ from 5-degenerate graphs from the viewpoint of broadcasting problem. The algorithm also gives a partial approximation to open problem (stated in [11]) whether there is a polynomial algorithm producing a broadcast schedule of the length $O(opt(G). \log n)$, where $opt(G)$ is the length of the shortest broadcast scheme on G. However the problem, whether it is possible to use an assumption about bounded average degree of arbitrary reachability graph in order to design algorithm producing shorter radio broadcast schedules, remains open.

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