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Note

A note on the lower bound of centralized radio broadcasting for planar reachability graphs

F. Galčík*

Institute of Computer Science, P.J. Šafárik University, Faculty of Science, Jesenná 5, 041 54 Košice, Slovak Republic

A R T I C L E I N F O

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A B S T R A C T

We consider deterministic radio broadcasting in radio networks whose nodes have full topological information about the network. We focus on radio networks having an underlying reachability graph planar. We show that there are reachability graphs such that it is impossible to complete broadcasting in less than 2.ecc(s, G) rounds, where ecc(s, G) is the eccentricity of a distinguished source s of a graph G. They provide a negative answer to the open problem stated by Manne, Wanq, and Xin in [F. Manne, S. Wanq, Q. Xin, Faster radio broadcasting in planar graphs, in: Proceedings of the 4th Annual Conference on Wireless on Demand Network Systems and Services, WONS'2007, IEEE Press, 2007, pp. 9–13] and show that it is impossible to complete the broadcasting task in $D + O(1)$ rounds for each planar reachability graph. Particularly, we propose $3/2.D$ lower bound with respect to the parameter D — the diameter of a reachability graph. It is a nontrivial lower bound of time of centralized radio broadcasting in the case that an underlying reachability graph is planar. Moreover, we describe a generalized construction which can potentially improve the presented lower bound.

1. Introduction

A *radio network* is a collection of autonomous stations that are referred to as *nodes*. The nodes communicate via sending messages. Each node is able to receive and transmit messages, but it can transmit messages only to the nodes, which are located within its transmission range. The network can be modelled by a directed graph called a *reachability graph* $G = (V, E)$. The vertex set of G consists of the nodes of the network and two vertices $u, v \in V$ are connected by an edge $e = (u, v)$, if and only if, the transmission of the node u can reach the node v. In such a case, the node u is called a *neighbor* of the node v. If the transmission power of all nodes is the same, then the reachability graph is symmetric, i.e. a symmetric radio network can be modelled by an undirected graph. In what follows, we consider only radio networks with undirected planar reachability graphs.

Nodes of the network work in synchronized steps (time slots) called *rounds*. Radio networks differ from other communication networks in the way how the nodes send and receive messages. In each round, a node can act either as a *receiver* or as a *transmitter*. A node u acting as a transmitter sends a message that can be potentially received by each of its neighbors. In a given round, a node, acting as a receiver, receives a message only if it has exactly one transmitting neighbor. The received message is the same as the message transmitted by the transmitting neighbor. If in a given round, a node u has at least two transmitting neighbors, then no message is received, and we say that a *collision* occurs at node u. It is also assumed that a node can determine its behavior in the following round within the actual round.

The goal of *broadcasting* is to disseminate a message from one distinguished node, called a *source*, to all other nodes. Remote nodes of the network are informed via intermediate nodes. For a given network and source, a *radio broadcast*

* Fax: +421 55 6222124.
E-mail address: frantisek.galcik@upjs.sk.

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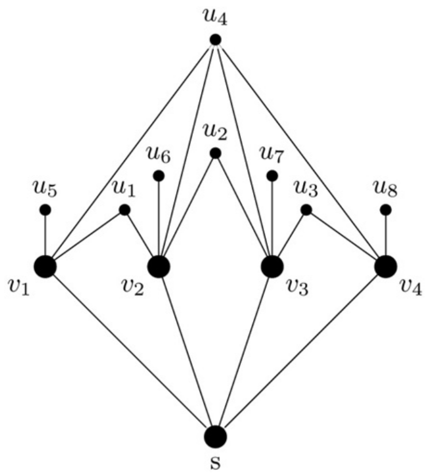

Fig. 1. Graph G_1.

schedule prescribes a set of informed nodes which have to transmit in a given round. The length of the schedule corresponds to the time required to complete the broadcast operation, i.e. to inform all nodes of the network. The time, that is required to complete an operation, is an important efficiency measure and is a widely studied parameter of mostly every communication task.

In this note, we focus on deterministic broadcasting in radio networks whose nodes have full topological knowledge of the network. Such a model is appropriate for radio networks with a stable topology. Since the network topology is known to all nodes, they can compute a radio broadcast schedule in advance. We consider only a subclass of radio networks — radio networks with planar reachability graphs. Results in [5,12] show the possibility of extremely fast completion of the radio broadcasting task in this subclass of radio networks, i.e. planar reachability graphs are radio broadcast friendly. When building a static radio network, we can utilize this fact, and choose locations of devices and directions of antennas in such a way that the resulting reachability graph will be planar. In a non-static setting, the nodes can begin to use fast broadcasting algorithms developed for planar graphs after they recognize (in polynomial time) that the underlying reachability graph is planar.

1.1. Related work

Information dissemination is studied as one of the basic communication primitives in all models of communication networks. In [6], a survey is given of known results related to the broadcasting task under communication modes investigated in the literature. In [9], the authors showed that the computation of the minimal time required to complete the broadcasting task is a *NP*-complete problem even for planar communication graphs. Note that the considered mode of radio communication differs significantly from other investigated modes. For instance, it is not possible to achieve broadcasting in less than $\Omega(\log n)$ rounds for any n-node network assuming a mode, when a node can communicate with $O(1)$-bounded number of neighboring nodes in a given round (e.g. the one-way mode, the two-way mode, or the mode investigated in [3]). On the other hand, there are radio networks in which the broadcasting task can be completed in $O(D)$ rounds, where D is the diameter of an underlying reachability graph that is independent of the number of nodes.

The study of communication in known topology radio networks [13] was initiated in the context of the broadcasting problem. In [7], the authors presented a deterministic polynomial-time algorithm producing a broadcast schedule of the length $O(D.\log^2 n)$ for any graph with n nodes and diameter D. The lower bound $\Omega(\log^2 n)$ of the length of a broadcasting schedule was proved in [1] for a family of graphs with diameter 2. Later in [4], the authors proposed a method improving the time of broadcasting in the case when a reachability graph is undirected. The method is based on partitioning of an underlying reachability graph into clusters with smaller diameter and applying broadcast schedules produced by known algorithms in each cluster separately. This method was later improved in [2]. Applying a deterministic algorithm from [10], which produces a broadcast schedule of the length $O(D.\log n + \log^2 n)$, method from [2] computes a broadcast schedule of the length $O(D+\log^4 n)$. Very recently in [5], these results have been further improved to $D+O(\log^3 n)$ for undirected graphs. Finally in [11], the authors proposed an algorithm producing a radio broadcast schedule of the asymptotically optimal length $O(D + \log^2 n)$.

In the case when an underlying reachability graph is undirected and planar, centralized broadcasting was first investigated in [2], where an algorithm producing schedules of the length $O(D + \log^4 n)$ was proposed. In [5] the authors showed an algorithm producing schedules of the length $3.D$. Very recently, the algorithm producing schedules of the length $D + O(\log n)$ was given in [12]. Note that in the previously mentioned results, parameter D can be replaced with the eccentricity of the source node $\mathrm{ecc}(s, G)$.

Recently, the authors of [8] proposed alternative models of radio networks (based on adversarial models) with the relaxed assumption of the network synchrony. They focused on centralized broadcasting and presented protocols with exponential work in the worst case.

1.2. Terminology and preliminaries

We use the standard graph terminology and notation. Particularly, the *distance* of two nodes u, v (denoted by $\text{dist}(u, v)$) is the length of a shortest $u - v$-path in the underlying reachability graph. The *eccentricity* of a node v in the graph G is defined as $\text{ecc}(v, G) = \max\{\text{dist}(v, u) | u \in V(G)\}$ and the *diameter* of the graph G as $D = \max\{\text{dist}(u, v) | u, v \in V(G)\}$.

The minimal number of rounds necessary to complete the broadcasting task by an optimal schedule with respect to the source s and the reachability graph G we shall denote as $\text{mintime}(s, G)$.

2. Lower bound of centralized broadcasting for the planar case

Utilizing algorithm from [5], the centralized broadcasting can be always completed in at most $3.\text{ecc}(s, G) - 2$ rounds in radio networks with planar reachability graphs, i.e. $\text{mintime}(s, G) \leq 3.\text{ecc}(s, G) - 2$ for each planar graph G. Obviously, broadcasting cannot be completed in less than $\text{ecc}(s, G)$ rounds, since $\text{ecc}(s, G)$ is the minimal distance between the source s and the most remote node. In this note, we show that there are undirected planar reachability graphs such that broadcasting in less than $2.\text{ecc}(s, G)$ rounds is not possible. Particularly, we present a sequence of planar graphs $\{G_k | k \geq 1\}$, such that there is a node $s \in V(G_k)$ satisfying

(1) $\text{ecc}(s, G_k) = 2.k$
(2) for each radio broadcast schedule with respect to the source s and the reachability graph G_k, there is a node that is firstly informed in the round $4.k = 2.\text{ecc}(s, G_k)$, i.e. $\text{mintime}(s, G_k) \geq 2.\text{ecc}(s, G_k)$
(3) $\text{ecc}(s, G_k) = \Theta(\log |V(G_k)|)$.

2.1. Construction of G_k

The construction of the sequence $\{G_k | k \geq 1\}$ is recursive. The (basic) graph G_1 (Fig. 1) is defined as follows

- $V(G_1) = \{s\} \cup \{v_i | i = 1, \ldots, 4\} \cup \{u_i | i = 1, \ldots, 8\}$
- $E(G_1) = \{(s, v_i), (v_i, u_4), (v_i, u_{i+4}) | i = 1, 2, 3, 4\} \cup \{(u_i, v_i), (u_i, v_{i+1}) | i = 1, 2, 3\}$.

In order to construct the graph G_k, for $k \geq 2$, we glue a copy of the graph G_{k-1} to each node u_i ($i = 1, \ldots, 8$) of the graph G_1 in such a way that $u_i = s$ where s is the source of G_{k-1}.

Clearly, G_k is a planar graph with the eccentricity of the distinguished source node s equal to $2.k = \text{ecc}(s, G_k)$. Since $|V(G_k)| = 8.|V(G_{k-1})| + 5$, for $k \geq 2$, it holds $\text{ecc}(s, G_k) = \Theta(\log |V(G_k)|)$.

2.2. Proof of the lower bound of centralized broadcasting

Now, we show that for each graph G_k and for each schedule of radio broadcasting with respect to the source s and the reachability graph G_k, there is a node $w \in V(G_k)$ such that w is firstly informed in the round $4.k = 2.\text{ecc}(s, G_k)$.

Theorem 1. *For each $k \geq 1$, any radio broadcast schedule for the graph G_k with respect to the source s has the length at least $2.\text{ecc}(s, G_k)$ rounds, i.e.* $\text{mintime}(s, G_k) \geq 4.k = 2.\text{ecc}(s, G_k)$.

Proof. At first, we consider the graph G_1 and show, that for any schedule there must exist a node that is uninformed after 3 rounds. Let us assume an optimal schedule. After the first round, all neighbors of the source are informed. Since the node u_4 has 4 neighbors, it becomes informed only after a round in which exactly one of its neighbor (v_1, v_2, v_3, or v_4) transmits. Without loss of generality, let the second round be a round after which the node u_4 is informed. After this round, there are 3 uninformed nodes of degree 1 and at least one uninformed node of degree 2. Note that nodes of degree 1 have distinct neighbors. Moreover, all neighboring nodes of uninformed nodes of degree 2 have an uninformed neighbor of degree 1. Thus, it is not possible to inform all nodes after the third round, since in order to inform uninformed nodes of degree 1, all their neighbors have to transmit. However, it causes a collision for uninformed nodes of degree 2.

Consider now the graph G_k, $k \geq 2$, and assume that according to the inductive hypothesis the claim is true for G_{k-1}. In the recursive construction of G_k, we can observe that the broadcasting in a copy of G_{k-1} is started after the corresponding node u_i of the 'basic' G_1 becomes informed. Broadcastings in different copies of G_{k-1} are independent, i.e. broadcasting in one copy has no influence on the broadcasting process in another copy. Claim for $k = 1$ implies the existence of a node u_i which becomes informed firstly in the round 4. Considering the assertion for the corresponding copy of G_{k-1}, we obtain $\text{mintime}(s, G_k) \geq 4 + \text{mintime}(u_i, G_{k-1}) \geq 4 + 4.(k-1) \geq 4.k$. □

2.3. Lower bound of centralized broadcasting with respect to diameter

Typically, the complexity of algorithms for centralized radio broadcasting (the length of a produced radio broadcast schedule) is described by an expression containing the parameter D — the diameter of a reachability graph. However, the parameter D can be replaced with the eccentricity of a source node $\mathrm{ecc}(s, G)$ in known algorithms for the planar case. It is easy to see that the constructed sequence of graphs $\{G_k | k \geq 1\}$ does not provide a better lower bound than the trivial lower bound D. Indeed, $D = 2.\mathrm{ecc}(s, G_k) \leq \mathrm{mintime}(s, G_k)$, for each $k \geq 1$. In fact, $2.\mathrm{ecc}(s, G_k) = \mathrm{mintime}(s, G_k)$. Using a simple extension of G_k, one can construct a better lower bound with respect to the parameter D. Let $\overline{G_k}$ be a planar graph constructed from G_k as follows: we add a path of the length $\mathrm{ecc}(s, G_k)$ starting in the source s. Let $\bar{s}$ be the last node of the added path. Clearly, diameter of the reachability graph is preserved, i.e. $D_{\overline{G_k}} = 2.\mathrm{ecc}(s, G_k) = D_{G_k} = \mathrm{ecc}(\bar{s}, \overline{G_k})$. Assume that $\bar{s}$ is the source of the broadcasting. Since each broadcasting have to follow a path from $\bar{s}$ to s of the length $\mathrm{ecc}(s, G_k)$, we obtain $\mathrm{mintime}(\bar{s}, \overline{G_k}) = \mathrm{ecc}(s, G_k) + \mathrm{mintime}(s, G_k) \geq 3.\mathrm{ecc}(s, G_k)$. Hence, $\mathrm{mintime}(\bar{s}, \overline{G_k}) \geq 3/2.D$.

Theorem 2. *For each $k \geq 1$, any radio broadcast schedule for the graph $\overline{G_k}$ with respect to the source s has the length at least $3/2.D$ rounds, i.e.* $\mathrm{mintime}(\bar{s}, \overline{G_k}) \geq 3/2.D$, *where D is diameter of $\overline{G_k}$.*

3. Generalized construction

In this section we generalize construction described in the previous sections. Let G be an undirected planar graph and denote

$$\mathrm{ratio}(G) = \max\left\{\frac{\mathrm{mintime}(v, G)}{\mathrm{ecc}(v, G)} | v \in V(G)\right\}.$$

For instance, we can consider the graph sequence defined in the previous section. Due to the inequality $\mathrm{mintime}(s, G_k) \geq 2.\mathrm{ecc}(s, G_k)$, for each G_k, $k \geq 1$, it follows that $\mathrm{ratio}(G_k) \geq 2$.

Let $s' \in V(G)$ be a node such that $\mathrm{ratio}(G) = \frac{\mathrm{mintime}(s', G)}{\mathrm{ecc}(s', G)}$. We construct a sequence of planar graphs $\{G'_k | k \geq 1\}$, such that there is a node $s' \in V(G'_k)$ satisfying

(1) $\mathrm{ecc}(s', G'_k) = k.\mathrm{ecc}(s', G)$
(2) for each radio broadcast schedule with respect to the source s' and reachability graph G'_k, there is a node that is firstly informed in the round $k.\mathrm{mintime}(s', G) = k.\mathrm{ratio}(G).\mathrm{ecc}(s', G) = \mathrm{ratio}(G).\mathrm{ecc}(s', G'_k)$
(3) $\mathrm{ecc}(s', G'_k) = \Theta(\log |V(G'_k)|)$.

Define $G'_1 = G$. For $k \geq 2$, G'_k is constructed recursively from G'_1 and G'_{k-1}. As in the previous section, we glue a copy of the graph G'_{k-1} to each node w of the graph G'_1, $w \neq s'$, in such a way that the node w is identified with the source s' of G'_{k-1}. Note that the construction preserves planarity (applying topological isomorphism, there is always a planar embedding of G' with the node s' on the outer face). Using similar argumentation, one can show by induction that all desired properties of G'_k according with respect to s' are valid. Finally, it is easy to see that the properties imply $\mathrm{ratio}(G'_k) \geq \mathrm{ratio}(G)$, i.e. it is not possible to complete radio broadcasting in less that $\mathrm{ratio}(G).\mathrm{ecc}(s', G'_k)$ rounds with respect to the source s' and the reachability graph G'_k.

Theorem 3. *Let G be an undirected planar graph. There exists a sequence of planar graphs $\{G'_k | k \geq 1\}$ such that for each $k \geq 1$, G'_k contains a node s' such that* $\mathrm{mintime}(s', G'_k) \geq \mathrm{ratio}(G).\mathrm{ecc}(s', G'_k)$.

Let $\overline{G'_k}$ be a planar graph constructed in such a way that we add a path of the length $\mathrm{ecc}(s', G'_k)$, which starts in the node s', to the graph G'_k. Let $\bar{s}$ be the last node of the added path. Considering $\bar{s}$ as a source of broadcasting, one can prove the following theorem providing generalized lower bound with respect to diameter of a graph.

Theorem 4. *Let G be an undirected planar graph. There exists a sequence of planar graphs $\{\overline{G'_k} | k \geq 1\}$ such that for each $k \geq 1$, $\overline{G'_k}$ contains a node $\bar{s}$ such that* $\mathrm{mintime}(\bar{s}, \overline{G'_k}) \geq (1 + \mathrm{ratio}(G))/2.D$, *where D is diameter of the graph $\overline{G'_k}$.*

4. Conclusion

Algorithm from [5] for centralized radio broadcasting which completes broadcasting in $3.\mathrm{ecc}(s, G) - 2$ rounds, implies that $\mathrm{ratio}(G) \leq 3 - \frac{2}{\mathrm{ecc}(s,G)}$ for each planar graph G. The construction presented in this note implies that there is an infinite sequence of planar graphs satisfying $\mathrm{ratio}(G) \geq 2$. Also, the generalized construction shows that in order to get a better lower bound of time of the centralized radio broadcasting for the planar case, we have to search for a planar graph with larger $\mathrm{ratio}(G)$ (in the best case with $\mathrm{ratio}(G) = 3 - \frac{2}{\mathrm{ecc}(s,G)}$).

The presented lower bounds, $2.\mathrm{ecc}(s, G)$ rounds and $3/2.D$ rounds, give a negative answer to the open problem stated in [12]. Particularly, they show that it is impossible to complete broadcasting in $D + O(1)$ rounds for each planar reachability graph. Another interesting open question is what bounds (upper or lower) can be achieved for planar graphs with $D = o(\log n)$.

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