On Siamese Color Graphs

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Abstract

A Siamese color graph is an edge decomposition of a complete graph into strongly regular subgraphs sharing a spread. Using a computer aided exhaustive search we complete the classification of Siamese color graphs on 40 vertices.

Keywords

Siamese color graphs, graph decompositions, generalized quadrangles, strongly regular graphs

1. Introduction

Siamese color graphs were initially introduced by Kharaghani and Torabi in [1] using algebraic methods and later studied by Klin, Reichard, and Woldar in [2, 3] from the geometric point of view. Kharaghani and Torabi provided an infinite class of Siamese color graphs arising from an infinite class of balanced generalized weighing matrices supplied by Gibbons and Mathon in [4]. Klin, Reichard, and Woldar presented a complete list of Siamese color graphs on 15 vertices and some geometric Siamese color graphs on 40 vertices [2, 3]. Last year we completed the classification of geometric Siamese color graphs on 40 vertices in [5]. Our current aim is to classify the pseudo-geometric and mixed cases, thereby complete the classification of Siamese color graphs on 40 vertices.

2. Preliminaries

2.1. Partial geometries and strongly regular graphs

A *partial geometry* is defined as an incidence structure characterized by the parameters (K, R, T). In this structure, each block (or line) includes K points, each point is on R lines, any pair of distinct points lies on at most one line, and for any line l and point p not on l, there are exactly T lines through p that intersect l.

By employing double-counting, it becomes evident that such a structure comprises $K\left({}^{(K-1)(R-1)}/_{T}+1\right)$ points and $R\left({}^{(K-1)(R-1)}/_{T}+1\right)$ lines.

A related concept to an incidence structure is its *point graph* (or *collinearity graph*), which vertices represent the points of the incidence structure, and two vertices are connected by an edge if and only if they lie on the same line.

A *strongly regular graph* with parameters (v, k, λ, μ) is defined as a regular graph with order v and valency $0 < k < v-1$, where each pair of adjacent vertices has λ common neighbors, and each pair of non-adjacent vertices has μ common neighbors.

It can be demonstrated that the point graph of any partial geometry is strongly regular with parameters:

$$\begin{aligned} v &= K\left(\frac{(K-1)(R-1)}{T}+1\right), \\ k &= (K-1)R, \\ \lambda &= (K-2)+(R-1)(T-1), \\ \mu &= RT. \end{aligned}$$

Conversely, a strongly regular graph that is the point graph of a suitable partial geometry is termed *geometric*. The *pseudo-geometric* strongly regular graph has the same parameter set as a geometric one, but does not come as a point graph of a given partial geometry.

A *spread* in a partial geometry is a set of pairwise disjoint lines that collectively cover all the points of the geometry. Since a spread partitions the $K\left({}^{(K-1)(R-1)}/_{T}+1\right)$ points of the partial geometry into disjoint sets of K points, there are ${}^{K((K-1)(R-1)/T+1)}/_{K}$ lines in a spread.

In the point graph, any two points on the same line in the spread are adjacent, thus forming a clique. If a spread is present in a partial geometry, it partitions the point set into $\frac{v}{K}$ cliques of size K. Consequently, any graph G with a disjoint set of equal-sized cliques spanning G is said to have a spread.

Consider a spread S in a partial geometry with parameters (K, R, T). For any distinct lines l and m in S, and any point p on m, exactly T lines through p intersect l at T distinct points. Similarly, for any point q on l, exactly T lines through q intersect m at T distinct points. Therefore, precisely KT lines intersect both l and m, intersecting in distinct pairs of points on l and m.

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Lemma 1. *Let (K, R, T) be a partial geometry with a spread. Then, for any two lines l and m in the spread, there are exactly KT other lines that intersect both. Each point on l is contained in exactly T of these lines.*

A (finite) generalized quadrangle with parameters (s, t) is an incidence structure W satisfying:

1. Each point is incident with $t + 1$ lines ($t \geq 1$), and two distinct points are incident with at most one line.

2. Each line is incident with $s + 1$ points ($s \geq 1$), and two distinct lines are incident with at most one point.

3. For any point x and line l not incident with x, there is exactly one line through x that intersects l.

The pair (s, t) is called the order of W. Hereinafter, a generalized quadrangle of order (s, t) is referred to as $GQ(s, t)$.

Generalized quadrangles are a specific case of partial geometries. Specifically, each generalized quadrangle of order (s, t) corresponds to a partial geometry with parameters $(s + 1, t + 1, 1)$.

Thus, the point graph of a generalized quadrangle of order (s, t) is a strongly regular graph with parameters:

$$
\begin{aligned}
v &= (s + 1)(st + 1), \\
k &= s(t + 1), \\
\lambda &= s - 1, \\
\mu &= t + 1.
\end{aligned}
$$

Each line in $GQ(s, t)$ forms a clique of size $1+s$ in the point graph. There are no other cliques due to the third condition in the definition of generalized quadrangles, which ensures that any three points inducing a K_3 in the point graph must lie on the same line.

There is a one-to-one correspondence between spreads in $GQ(s, t)$ and spreads in its point graph consisting of $1 + st$ cliques of size $1 + s$.

By Lemma 1 we have that any two cliques in the spread are connected by exactly $1 + s$ edges, forming a perfect matching. This is articulated as follows:

Lemma 2. *If the vertices of the point graph of $GQ(s, t)$ with a spread S are arranged according to their corresponding cliques, the resulting adjacency matrix can be represented by $(1 + st) \times (1 + st)$ blocks, each of size $(1 + s) \times (1 + s)$. The diagonal blocks of the matrix correspond to the adjacency matrices of the cliques (i.e., $J_{1+s} - I_{1+s}$), while the off-diagonal blocks are permutation matrices, representing the incidence matrices of 1-factors.*

A graph G with diameter d is distance-regular if and only if there is an array of integers $(b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d)$ such that for all $1 \leq j \leq d$, and any pair of vertices u and v at distance j in G, b_j gives the number of neighbors of u at distance $j + 1$ from v, and c_j gives the number of neighbors of u at distance $j - 1$ from v. This array of integers is known as the intersection array of a distance-regular graph.

As demonstrated by Brouwer in [6], removing the edges of a spread S from a strongly regular graph G with parameters given by the point graph of $GQ(s, t)$ results in a distance-regular graph of diameter 3 with an antipodal system S. In this system, the relation of being at distance 3 in the distance-regular graph $G - S$ is an equivalence relation, with blocks corresponding to the cliques of S. Hence Lemma 2 can also be applied to pseudo-geometric strongly regular graph with spread.

Lemma 3. *Let G be a pseudo-geometric or geometric strongly regular graph with a spread S consisting of n cliques of size k. If the vertices of G are arranged according to their corresponding cliques in S, the resulting adjacency matrix can be represented by $n \times n$ blocks, each of size $k \times k$. The diagonal blocks of the matrix correspond to the adjacency matrices of the cliques (i.e., $J_k - I_k$), while the off-diagonal blocks are permutation matrices, representing the incidence matrices of 1-factors.*

If the strongly regular graph G is geometric or pseudo-geometric, the resulting distance-regular graph $G - S$ is termed *geometric* or *pseudo-geometric*, respectively.

2.2. Siamese color graphs

A *color graph* Γ is defined as a pair $(V, \mathcal{R})$ where V is a set of vertices and $\mathcal{R}$ is a partition of V^2, meaning that the elements of $\mathcal{R}$ are pairwise disjoint and $\bigcup_{R \in \mathcal{R}} R = V^2$. The relations in $\mathcal{R}$ are referred to as the colors of Γ, and the number $|\mathcal{R}|$ of its colors is called the rank of Γ.

In other words, a color graph is any edge coloring of a complete digraph with a loop at each vertex. We define an *adjacency matrix* of a color graph to be a $v \times v$ matrix $A = (a_{i,j})$ such that $a_{i,j} = t$ if $(x_i, x_j) \in R_t$ for $R_t \in \mathcal{R}$.

Throughout this paper, we will consider only color graphs where all their colors represent symmetric relations, i.e., underlying graphs of non-trivial relations are simple and undirected, and one of them is the identity relation..

Let Γ and Γ' be color graphs. An isomorphism $\phi : \Gamma \to \Gamma'$ is a bijection of V onto V' which induces a bijection $\psi : \mathcal{R} \to \mathcal{R}'$ of colors. A weak (or color) automorphism of Γ is an isomorphism $\phi : \Gamma \to \Gamma$. If, in addition, the induced map ψ is the identity on $\mathcal{R}$, we call ϕ a (strong) automorphism of Γ.

In 2003, Kharaghani and Torabi introduced the concept of a Siamese color graph, i.e., the decomposition of a complete graph into strongly regular graphs sharing a spread. This notion is formalized in the following definition.

Definition 4. Let $W = (V, \{Id_V, S, R_1, R_2, \ldots, R_n\})$ be a color graph such that

1. (V, S) is a partition of V into cliques of equal size.
2. For all i, the graph (V, R_i) is an imprimitive distance-regular graph of diameter 3 with antipodal system S.
3. For all i, the graph $(V, R_i \cup S)$ is a strongly regular graph with fixed parameters.

Then W is a *Siamese color graph.* We call S the spread of Γ and n — the number of distance-regular graphs — the Siamese rank of W.

We shall denote W by $SCG(v, k, \lambda, \mu, \sigma)$ where (v, k, λ, μ) are the common parameters of all $srg(V, R_i \cup S)$ and σ is the valency of the spread S. Kharaghani and Torabi used the term *Siamese* to indicate that all these strongly regular graphs share a common spread.

Moreover, Kharaghani and Torabi in [1] proved the existence of an infinite family of Siamese color graphs with special parameters.

Theorem 5. *For any prime power q, there exists an $SCG(1+q+q^2+q^3, q+q^2, -1+q, 1+q, q)$, which is an SCG on $1+q+q^2+q^3$ vertices consisting of $1+q$ strongly regular graphs sharing $1+q^2$ disjoint cliques of size $1+q$.*

The parameters of the strongly regular graphs mentioned above are of interest as they match those of a point graph of a generalized quadrangle $GQ(q, q)$. Hereafter, Siamese color graphs with these parameters are referred to as Siamese color graphs of *order* q, denoted as $SCG(q)$. According to Brouwer's Theorem [6], for this class of Siamese color graphs, the verification of the second condition in Definition 4 is unnecessary if the remaining two conditions are met.

A Siamese color graph $SCG(q)$ is termed *geometric* if all its strongly regular graphs $(V, R_i \cup S)$ are geometric, *pseudo-geometric* if all its strongly regular graphs $(V, R_i \cup S)$ are pseudo-geometric, and *mixed* if it contains both pseudo-geometric and geometric strongly regular graphs $(V, R_i \cup S)$.

3. Some Known Results on Siamese Color Graphs

Geometric Siamese color graphs were studied by Reichard in his thesis [7] and further by Klin, Reichard, and Woldar in a series of articles [2, 3]. In these papers, the authors constructed an infinite family of geometric Siamese color graphs, conjectured to be isomorphic to the family of Kharaghani and Torabi, and proved the following result.

Theorem 6 ([2, 3])**.** *Let W be a geometric Siamese color graph of order q. For each point graph $(V, R_i \cup S)$, construct the corresponding generalized quadrangle. Let B denote the union of all lines in all resulting generalized quadrangles. Then the incidence structure*

$$\mathcal{S} = (V, B)$$

is a Steiner design

$$\mathcal{S} = S\left(2, q+1, \frac{q^4-1}{q-1}\right).$$

Using Theorem 6, Klin, Reichard, and Woldar completely classified Siamese color graphs of order 2 and found hundreds of geometric Siamese color graphs of order 3. The classification of Siamese color graphs of order 2 was expressed in the following theorem.

Theorem 7 ([2, 3])**.** *Every Siamese color graph on 15 vertices is necessarily geometric. There are exactly two non-isomorphic Siamese color graphs on 15 vertices. Their corresponding Steiner triple systems are $STS(15)\#1$ and $STS(15)\#7$ in the notation of [8].*

Last year we contributed to this research in [5] by completing the classification of geometric Siamese color graphs of order 3 which can be summarised in the following theorem.

Theorem 8. *There are exactly 399 non-isomorphic Siamese color graphs on 40 vertices. Their corresponding Steiner systems $S(2, 4, 40)$ are all non-isomorphic.*

4. Siamese color graphs of order 3

According to the definition, Siamese color graphs must have a spread consisting of ten K_4 and include four strongly regular graphs with parameters $(40, 12, 2, 4)$. Among the 28 strongly regular graphs of order 40 [9, 10], only two posses a spread: one geometric — $\#26$ in [10] and one pseudo-geometric — $\#27$ in [10]. The geometric one shall be denoted G and pseudo-geometric P. For both of these graphs, the distance-regular graphs have intersection arrays $\{9, 6, 1; 1, 2, 9\}$. The case of a purely geometric Siamese color graph was resolved in [5]. However, as we seek to classify both purely pseudo-geometric Siamese color graphs and the mixed case, we will provide a characterisation of both types of strongly regular graphs and their distance-regular graphs.

4.1. Strongly regular and distance-regular graphs with spread on 40 vertices

The geometric strongly regular graph G is the point graph of $GQ(3,3)$. It contains 40 cliques of size 4. It possesses 36 different spreads, all of which can be mapped onto each other under the group of automorphisms. Its group of automorphisms comprises 51 840 elements, with the stabilizer of the spread having 1 440 elements. However, there is only one non-trivial automorphism that also fixes the order of cliques in the spread. Distance-regular graph of G will be denoted G_1.

The pseudo-geometric strongly regular graph P contains only 22 cliques of size 4, with only four possible spreads. One of these is stable under the group of automorphisms while other three map onto each other. Its group of automorphisms comprises 432 elements. We analyse this scenario in two distinct cases, especially since removing all edges of one of these spreads from the graph always results in one of two non-isomorphic distance-regular graphs.

In case of one stable spread, the stabilizer of the spread is identical to the group of automorphisms and there are three non-trivial automorphisms that also fix the order of cliques in the spread. The distance-regular graph of P is then denoted P_1.

In case of three isomorphic spreads, the stabilizer of the spread has 144 elements, and the only automorphism that fixes the order of cliques in spread is trivial e.g. the identity. The distance-regular graph of P is then denoted P_2.

4.2. (Siamese) twins and preferred twins

In what follows S will always be the spread on 40 vertices with cliques $\{1,2,3,4\},\{5,6,7,8\},\ldots,\{37,38,39,40\}$ and any distance regular graph will have parameters $\{9,6,1;1,2,9\}$ and antipodal system S. Graphs G_1, P_1, P_2 mentioned above will be lexicographically maximal elements of their orbits $O(G_1)$, $O(P_1)$ and $O(P_2)$ of $Aut(S)$, respectively.

We say that two distance regular graphs G and G' are *twins* if they are edge-disjoint.

During our computations we found all twins G, G' such that G is one of the above mentioned graphs G_1, P_1, P_2 and G' is isomorphic to one of them. Numbers of possible twins are summarised in the following table.

$G\backslash G'$	G_1	P_1	P_2
G_1	1 244 574	11 247 360	2 880
P_1	3 374 208	33 567 444	18 144
P_2	288	6 048	4 608

Graphs are ordered by the size of their automorphism group ($|Aut(G_1)| = 1440$, $|Aut(P_1)| = 432$, $|Aut(P_2)| = 144$). Let us note that elements of our table comply with the following lemma:

Lemma 9. *Let $n_{i,j}$ denote the number of graphs in $\Gamma_j^{Aut(S)}$, which are edge-disjoint with Γ_i. Then $n_{i,j} \cdot |Aut(\Gamma_j)| = n_{j,i} \cdot |Aut(\Gamma_i)|$.*

Furthermore, we observed that some instances of distance-regular graphs form twins with the three aforementioned distance-regular graphs in a significantly better way than others. Specifically, they produce a much greater number of Siamese color graphs that contain them compared to the relatively small number of graphs that do not contain any such distance regular graphs. A notion of *preferred twin* is introduced to describe this surprising phenomenon in our results. A twin Γ' of Γ is defined as a preferred twin if there exists a further division of vertices in the cliques of the spread into pairs such that the mapping $\varphi : E(G) \to E(G)$, which exchanges vertices in these pairs, is a fixed-point-free involution and an automorphism on G. The preferred twin G' is a graph on the same set of vertices as G, with edges such that $(a,b) \in E(G) \Leftrightarrow (a,\varphi(b)) \in E(G')$. This mapping φ shall be called a *preferred pairing*.

It can be shown that G' is a strongly regular graph with the same set of parameters as G. All the preferred twins found in our computations consist of two isomorphic distance-regular graphs .

The graph G_1 has a unique preferred twin. Its automorphism group is identical to $Aut(G_1)$. There are three preferred twins for P_1, forming one orbit under the action of $Aut(P_1) \cap Aut(S)$. All the referred twins of P_1 are also preferred twins pair-wise. There is no preferred twin for P_2.

Remark 10. Let G and G' be preferred twins, φ be their preferred pairing, and M and M' be their adjacency matrices, respectively. Then M' can be derived from M by applying φ only to the rows or only to the columns of M. Hence, if the rows and columns within cliques are arranged so that rows exchanged by φ are adjacent, this further subdivides the blocks of size 4×4 in both M and M' into 2×2 blocks in such a way that each of them is either all-zero, I_2, or $J_2 - I_2$. Moreover, M and M' have all all-zero blocks in the same positions, and M has its I_2 blocks in the positions where M' has $J_2 - I_2$, and vice versa. This phenomenon is further illustrated by the following minor of the matrix $G_1 + 2 \times G_1'$.

$$\left[\begin{array}{c|cccc|cccc|cccc|c}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\hline
\cdots & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & \cdots \\
\hline
\cdots & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & \cdots \\
\cdots & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & \cdots \\
\cdots & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & \cdots \\
\hline
\cdots & 0 & 0 & 2 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & \cdots \\
\hline
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]$$

From Remark 10, it can be observed that for any preferred twins G, G', the complement of the graph $G + G' + S$ again appears as $H + H'$ for some preferred twins H, H'. Indeed, it can be easily verified that this is the case.

Lemma 11. *For any two pairs of preferred twins G, G' and H, H', it holds that $(\overline{S + G + G'}) \cong_S G + G' \cong_S H + H'$, where $\cong_S$ indicates that there exists an isomorphism belonging to $Aut(S)$.*

This also explains the existence of some mixed Siamese color graphs. Clearly, in any Siamese color graph with a preferred twin G, G', this twin can always be substituted by some other preferred twin H, H', for G not isomorphic to H and the result will again be a Siamese color graph. To explain the prevalence of Siamese color graphs with preferred twins, the following results will be useful.

Lemma 12. *Let G be a distance-regular graph and G' be its preferred twin. Then $|Aut(G + G')| = 2^{20} \times 1440$ and $Aut(G + G') \leq Aut(S)$.*

Lemma 13. *Let $W = (V, \{Id_V, S, G, G', G'', G'''\})$ be a Siamese color graph with preferred twins G, G', and let $\psi \in Aut(G + G')$. Then $W' = (V, \{Id_V, S, G, G', \psi(G''), \psi(G''')\})$ is also a Siamese color graph.*

Remark 14. The size of the automorphism group for any Siamese color graph is at most $4 \times (Aut(G) \cap Aut(S)) = 5760$ for $G \in \{G_1, P_1, P_2\}$, as there are at most 4 possible ways to map 4 distance-regular graphs in a Siamese color graph onto each other. Hence for there is at most 5760 maps that return the same Siamese color graph. Number of different but isomorphic Siamese color graphs containing given preferred twins G, G' is at most $4 \times Aut(G)$ (if all distance-regular graphs in Siamese color graph are isomorphic). Compared with the size of $Aut(G + G')$ from Lemma 12, and using Lemma 13 and the pigeonhole principle, there are at least $|Aut(G + G')|/|4 \times (Aut(G) \cap Aut(S)) \times 4 \times Aut(G)| \geq (2^{20} \times 1440)/(4 \times 1440 \times 4 \times 1440) = 2^{16}/1440 \sim 45.5$ non-isomorphic Siamese color graphs containing the pair G, G' of preferred twins.

However, the bound on the size of the group of automorphisms mentioned in Remark 14 is achieved only for one geometric and one pseudo-geometric Siamese color graph, and it is usually much smaller. Hence the number of Siamese color graphs containing any given preferred pair up to isomorphism is actually much larger.

5. Computer-aided search

Although our primary emphasis in the search was on pseudo-geometric Siamese color graphs of order 3, the mixed case was also examined. The classification of pseudo-geometric and mixed Siamese color graphs is computationally more difficult in comparison to the classification of geometric Siamese color graphs in [5]. We succeeded thanks to significant improvements made in the original program.

One of the main differences compared to the purely geometric case is that there are two non-isomorphic pseudo-geometric distance-regular graphs. Therefore, for a fixed spread S, the pseudo-geometric distance-regular graphs with the antipodal system S form two distinct orbits under $Aut(S)$. For the sake of simplicity, the following four-step strategy was employed, focusing on Siamese color graphs containing four isomorphic distance-regular graphs, ensuring that all eligible distance-regular graphs are in the same orbit under $Aut(S)$. Strategies for Siamese color graphs containing non-isomorphic distance-regular graphs, either geometric or pseudo-geometric, will be considered in subsequent sections.

5.1. Computer-aided search for pseudo-geometric Siamese color graphs of order 3

Naive approach:

1. For a fixed spread S, a pseudo-geometric distance-regular graph Γ_1 with the antipodal system S is chosen.
2. All automorphisms of S are applied to obtain all pseudo-geometric distance-regular graphs with S as the antipodal system, and the set A is found containing all such distance-regular graphs that have no common edges with Γ_1.
3. Triples Γ_2, Γ_3, Γ_4 of mutually edge-disjoint distance-regular graphs are identified in A.
4. The resulting system of Siamese color graphs is checked for isomorphism.

Various improvements were implemented to expedite computations. Here we introduce the most significant modifications made in each step.

Representation of distance-regular graphs : The spread S was fixed with cliques $\{1, \ldots, 4\}, \{5, \ldots, 8\},$

$\ldots, \{37, \ldots, 40\}$. This choice of S facilitated representation of graphs during computations by:

- **adjacency matrix** – 40×40 matrices that can be divided into 10×10 blocks of size 4×4. From the Lemma 3, off-diagonal blocks are permutation matrices. The distance-regular graphs have zero matrices in place of diagonal blocks – Step 1, 3
- **binary number representation** – numbers such that the binary AND of any two returns zero if and only if the distance-regular graphs they represent are edge-disjoint. These are generated by concatenating rows of each block matrix above the diagonal into one 16 digit binary number, and then concatenating these – Steps 2 and 3.
- **permutation matrix representation** – matrices 10×10 such that every entry represents a 4×4 block in their adjacency matrices. Since these blocks are either permutation matrices or all-zero, they can be represented by numbers in $\{0, 1, \ldots, 24\}$. Specifically, all-zero is represented by 0, and permutation matrices by $\{1, \ldots, 24\}$ in such a way that the greater the 16 digit number derived by concatenation of rows of a block, the smaller the entry in matrix representation, e.g., I is represented by 1 – Steps 1, 2, and 4.

The binary number representation allows us to compare individual graphs and to introduce an ordering both on A a set of all Siamese color graphs. Hereinafter the ordering of distance-regular graphs in A is given by the ordering of their binary representations. Siamese color graphs are represented by quadruple of their distance-regular graphs in descending order and ordered by this representation as well.

To avoid the repetitions of the Siamese color graphs in Step 3 we are striving to find only the greatest representation for each Siamese color graph.

Step 1:
As we are looking for greatest representation for each of the Siamese color graphs, Γ_1 was chosen as the graph with the greatest binary number representation. Consequently, all blocks beside the first one in the first row of the adjacency matrix of Γ_1 are I, and all entries except the first one of the permutation matrix representation are 1.

Step 2:
In order to obtain all images of Γ_1 in $Aut(S)$ the action of $Aut(S) = S_4 \wr S_{10} = S_4^{10} \rtimes S_{10}$ on the cliques of S was utilized in the following way. For each ϕ in S_{10}, an auxiliary graph $\Gamma' = \Gamma_1^{\phi}$ was constructed, and then a backtrack procedure was used to filter all $\overline{\psi} \in S_4^{10}$ such that Γ'^{ψ} is edge-disjoint with Γ_1.

To further accelerate the computation, the fact that $H = Aut(\Gamma_1) \leq Aut(S)$ was exploited. Specifically, only permutations ϕ from a transversal of $S_{10} = Aut(S)/S_4^{10}$ with respect to H/S_4^{10} were tested, and for a given ϕ, only $\overline{\psi}$ from different cosets of H in S_4^{10} were considered.

Step 3:
Graphs in the set A are sorted into sets by the second to fifth block (of size 4×4) in the first row of its adjacency matrix. We shall refer to these blocks as B_2, B_3, B_4, and B_5. The first block is zero, since Γ_i is a distance-regular graph. In Γ_1 we have $B_2 = B_3 = B_4 = B_5 = I_4$.

Firstly the graphs in the set A are sorted into three sets denoted A_2, A_3, and A_4, based on the position of 1 in the first row of B_2 in their adjacency matrices. Since Γ_1 has 1 at position $(1, 1)$, adjacency matrices of each graph in A must have 1 at exactly one of the positions $(1, i), i \in \{2, 3, 4\}$. The division of A into A_2, A_3, and A_4 follows naturally. It is easy to see, that for any triples $\Gamma_2, \Gamma_3, \Gamma_4$ of mutually edge-disjoint distance-regular graphs in A, no two Γ_i, Γ_j belong to the same A_k. Therefore, without the loss of generality we can assume that $\Gamma_i \in A_i$ for $i = 2, 3, 4$.

This is further subdivided into subsets by the blocks B_2, B_3, B_4, and B_5. There are only 4 combinations of three permutation matrices (of size 4×4) such that the sum of these matrices and an identity matrix yields an all-ones matrix. A list of all possible combinations of permutation matrices in the blocks B_2, B_3, B_4, and B_5 for $\Gamma_i, i = 2, 3, 4$ was made and the computations were distributed in such a way that in each instance we restricted candidates for Γ_2 and Γ_3 to graphs with the prescribed second to fifth block of the first row of the adjacency matrix.

Previously, when this step was executed for purely geometric Siamese color graphs, possible edge-disjoint pairs of Γ_2 and Γ_3 were checked by examining all combinations from two lists. This check involved simply applying AND to all possible pairs, as this returns 0 exactly when graphs are edge-disjoint. While this method sufficed for the purely geometric case, the purely pseudo-geometric case involved 30 times more possible Γ_2, Γ_3, and Γ_4 matrices. Consequently, comparing two lists would take on average 900 times longer, and this part of computation would take almost a month.

However, improvements were made by storing binary representations of possible Γ_2s in a prefix tree data structure. This class included a method that, given as an input a binary representation of Γ_3, yielded binary representations of all Γ_2 that were edge-disjoint to the given Γ_3. This enhancement significantly accelerated our program, reducing the computation time to less than an hour[1].

[1] We also ran it for the geometric case and thus have an independent verification of the results from [5]. This computation previously

Clearly, in each Siamese color graph of order 3, Γ_4 is uniquely determined by S, Γ_1, Γ_2, and Γ_3. It was found that for given edge-disjoint Γ_2 and Γ_3, it was faster to first check whether $K_{40} - (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ was a $srg(40, 12, 2, 4)$, and only then verify whether it belongs to set A. Instances of Γ_4 such that $\Gamma_4 + S$ is $srg(40, 12, 2, 4)$ but Γ_4 is not in A, would be also retained, as they would be useful in constructions of the mixed Siamese color graphs, however these instances did not occur.

Interestingly, both in the case of Siamese graphs consisting of four G_1 and of four P_1, one preferred twin of Γ_1 belongs to Γ_2 and it has the largest binary representation in the whole A. Since we are looking for greatest representation for each of the Siamese color graphs, we can divide our search into two cases. First being the case without the preferred twins, here we discard all the preferred twins of Γ_1 out of A and the rest is the same. In the later case, we start with preferred twins as Γ_1, Γ_2 and we can discard most of the possible Γ_3 in such a way, that we only check for those Γ_3, that have the largest binary representation in their orbit under the weak automorphism group of color graph with colors S, Γ_1 and Γ_2.

Step 4:
Instead of testing all obtained Siamese color graphs for isomorphisms, it suffices to check whether their binary representation is maximal in the action of $Aut(S)$. Graph Γ_1 already has a lexicographically maximal binary representation in its orbit under the action of $Aut(S)$. Therefore, for any Siamese color graph $W = (V, \{Id_V, S, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\})$, it is sufficient to select one map ψ_i such that $\psi_i(\Gamma_i) = \Gamma_1$ for all $i \in \{1, 2, 3, 4\}$, and then check all maps of the form $\phi\psi_i$ where $\phi \in Aut(\Gamma_1)$.

5.2. Mixed case

The search for mixed Siamese color graphs is very similar to the pure cases. We have to consider how many geometric and how many and which pseudo-geometric distance-regular graphs are to be in the final Siamese color graph and in what order are they to be assigned to Γ_1, Γ_2, Γ_3, and Γ_4. This was facilitated by including a simple step in the previous cases. In Step 3, we always first checked if $\Gamma_4 + S$ was strongly regular, and only afterward verified whether it belonged to set A. As a result of these computations we have that there are no mixed Siamese color graphs containing three isomorphic copies of a distance regular graph. Hence, a mixed Siamese regular graph contains at most two pairs. However, by the pigeonhole principle, there must be at least one pair of isomorphic distance-regular graphs in a mixed Siamese color graph.

It is straightforward to count that there are 6 possible combinations of distance-regular graphs: 3 combinations with two isomorphic pairs and 3 combinations with one isomorphic pair and one of each remaining distance-regular graphs. The remaining task is to assign the order in which these are assigned to Γ_1, Γ_2, Γ_3, and Γ_4. Clearly, for any mixed Siamese color graph and any distance-regular graph G it contains, there exists an isomorphic mixed Siamese color graph with Γ_1 isomorphic to G. In other words, for any combination, we can choose which distance-regular graph is going to be Γ_1. Furthermore, the stabilizer of P_1 as Γ_1 in $Aut(S)$ permutes the block B_2 in adjacency matrix in such a way that we can arbitrarily choose the order in which the other three distance-regular graphs are assigned to Γ_2, Γ_3, and Γ_4 without loss of generality. This flexibility allows us to solve all five cases containing P_1 without further case work. The only remaining case consists of two P_2s and two G_1s. There, the stabilizer of P_2 as Γ_1 in $Aut(S)$ permutes the first permutation block of adjacency matrices in such a way that we can choose which distance-regular graph will be Γ_2, and hence we can choose both Γ_1 and Γ_2 to be isomorphic to P_2, and so both Γ_3 and Γ_4 must be isomorphic to G_1.

Further differences lie in Steps 2 and 3, affecting the set A and its subsequent subdivision. In Step 2, the representatives of the cosets of H in $Aut(S)$ are not applied to Γ_1, but rather to those of G_1, P_1, and P_2 that were previously assigned to Γ_2, Γ_3, or Γ_4, yielding one set A for each of the considered G_1, P_1, and P_2. These sets are then further divided or filtered in Step 3 in an evident manner. In Step 4, ψ_i is only selected for those i for which Γ_i is isomorphic to Γ_1.

Our strategy was implemented using Python [11], GAP [12], and GAP packages GRAPE and DESIGN [13, 14].

6. Results

Let us begin with some preliminary results which may be of independent interest. In the Step 3, for Γ_1 isomorphic to P_2 there were no two graphs Γ_2 and Γ_3 with no common edges.

Lemma 15. *No Siamese color graphs contains P_2 as a distance-regular factors.*

Moreover, in the Step 3 for Γ_1, Γ_2 and Γ_3 being either all isomorphic to G_1 or all isomorphic to P_1, all possible Γ_4 were isomorphic to Γ_1 as well. Hence, if there is a Siamese color graph with three isomorphic distance regular graphs then the fourth is isomorphic to the others as well.

took about a day but now was finished in the matter of minutes.

Theorem 16. *Siamese color graphs of order 3 exist in three forms:*

- *geometric; all distance-regular factors are isomorphic to G_1,*
- *pseudo-geometric; all distance-regular factors are isomorphic to P_1*
- *mixed; two distance-regular factors are isomorphic to G_1 the rest is isomorphic to P_1.*

For the sake of completenesses we state the improved Theorem 8.

Theorem 8*. *There are exactly 399 non-isomorphic Siamese color graphs on 40 vertices, 357 of them consist of two preferred twins. Their corresponding Steiner systems $S(2, 4, 40)$ are all non-isomorphic.*

Theorem 17. *There are 20 354 pseudo-geometric Siamese color graphs, 20 030 of them consist of two preferred twins — in one instance any two of its distance-regular factors are preferred twins. Remaining 324 pseudo-geometric Siamese color graphs contain no preferred twins.*

Theorem 18. *There is 4 492 mixed Siamese color graphs, 4 480 of them consist of two preferred twins: one set of twins is geometric and the other is pseudo-geometric. The remaining 12 Siamese color graphs contain no preferred twins.*

For each of the 20 354 pseudo-geometric and 4 492 mixed Siamese color graphs of order 3, some algebraic properties were computed. These properties pertain to the given Siamese color graphs and a block design derived from them in a manner similar to Theorem 6. In the tables below, each row represents a group of non-isomorphic Siamese color graphs that otherwise share all computed properties. The computed properties for Siamese color graphs include the automorphism group (column marked as A(SCG)), its orbit on vertices (O(SCG)), and the weak automorphism group (WA(SCG)). For the block design derived from cliques of the Siamese color graph, the automorphism group (A(BD)) and its orbits on the blocks (O(BD)) were computed. To simplify the table, only the sizes of the identified groups and orbits are stated. The last column indicates how many of these non-isomorphic Siamese color graphs contain at least one preferred twin.

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Table 1: Pseudo-geometric Siamese color graphs for $q = 3$.

|A(SCG)|	O(SCG)	|A(BD)|	O(BD)	WA(SCG)	# preferred / #
72	$4, 36$	1728	$1, 9, 48$	1728	$1/1$
72	$4, 36$	1728	$1, 9, 48$	864	$0/1$
72	$4, 36$	1728	$1, 9, 48$	576	$1/1$
72	$4, 36$	1728	$1, 9, 48$	288	$1/1$
72	$4, 36$	1728	$1, 9, 48$	144	$0/1$
36	$2^2, 18^2$	576	$1, 9, 48$	288	$2/2$
36	$2^2, 18^2$	576	$1, 9, 48$	144	$0/1$
36	$2^2, 18^2$	576	$1, 9, 48$	144	$0/1$
36	$2^2, 18^2$	576	$1, 9, 48$	144	$0/1$
36	$2^2, 18^2$	576	$1, 9, 48$	72	$0/1$
24	$4, 12, 24$	72	$1, 3, 4, 6, 8, 36$	72	$0/2$
24	$4, 12, 24$	48	$1, 3, 4, 6, 8, 12, 24$	48	$4/4$
12	$2^2, 6^2, 12^2$	48	$1, 3, 4, 6, 8, 12, 24$	24	$2/2$
12	$2^2, 6^2, 12^2$	24	$1, 2^2, 3, 4^2, 6^3, 12^2$	24	$8/8$
12	$4, 12^3$	24	$1, 3, 4^3, 6, 12, 24$	24	$4/4$
8	$4^2, 8^4$	1728	$1, 9, 48$	192	$0/1$
8	$4^2, 8^4$	1728	$1, 9, 48$	96	$0/1$
8	$4^2, 8^4$	1728	$1, 9, 48$	64	$0/1$
8	$4^2, 8^4$	96	$1^2, 8, 16, 32$	96	$0/4$
8	$4^2, 8^4$	64	$1^2, 8, 16, 32$	64	$2/4$
8	$4^2, 8^4$	64	$1^2, 8, 16, 32$	32	$1/1$
8	$4^2, 8^4$	64	$2, 8, 16, 32$	64	$4/8$
8	$4^2, 8^4$	64	$2, 8, 16, 32$	32	$2/2$
8	$4^2, 8^4$	32	$1^2, 4^2, 8^2, 16^2$	32	$8/12$
8	$4^2, 8^4$	32	$2, 8, 16, 32$	32	$0/2$
8	$4^2, 8^4$	32	$1^2, 8, 16, 32$	32	$0/2$
8	$4^2, 8^4$	32	$2, 8, 16, 32$	32	$0/12$
8	$4^2, 8^4$	24	$1^2, 2, 4, 6, 8, 12, 24$	24	$0/4$
8	$4^2, 8^4$	16	$1^2, 2^2, 4^3, 8^3, 16$	16	$12/20$
8	$4^2, 8^4$	16	$2, 4^2, 8^2, 16^2$	16	$2/24$
8	$4^2, 8^4$	16	$2, 4^2, 8^2, 16^2$	8	$1/1$
8	$4^2, 8^4$	16	$1^2, 4^2, 8^2, 16^2$	16	$2/25$
8	$4^2, 8^4$	16	$1^2, 4^2, 8^2, 16^2$	8	$1/1$
8	$4^2, 8^4$	8	$1^2, 2^4, 4^4, 8^4$	8	$0/44$
6	$2^2, 6^6$	24	$1, 3, 4^3, 6, 12, 24$	12	$2/2$
4	$2^4, 4^8$	576	$1, 9, 48$	32	$0/1$
4	$2^4, 4^8$	576	$1, 9, 48$	16	$0/1$
4	$2^4, 4^8$	64	$1^2, 8, 16, 32$	32	$2/3$
4	$2^4, 4^8$	64	$2, 8, 16, 32$	16	$2/2$
4	$2^4, 4^8$	32	$1^2, 8^3, 16^2$	32	$8/10$
4	$2^4, 4^8$	32	$1^2, 8^3, 16^2$	16	$4/5$
4	$2^4, 4^8$	32	$1^2, 8^3, 16^2$	16	$0/2$
4	$2^4, 4^8$	32	$1^2, 4^2, 8^2, 16^2$	16	$4/5$
4	$2^4, 4^8$	16	$1^2, 8^3, 16^2$	16	$2/4$
4	$2^4, 4^8$	16	$1^2, 8^3, 16^2$	8	$1/1$
4	$2^4, 4^8$	16	$1^2, 2^2, 4^3, 8^3, 16$	8	$6/8$
4	$2^4, 4^8$	16	$1^2, 4^4, 8^3, 16$	8	$8/8$
4	$2^4, 4^8$	16	$1^2, 4^6, 8^4$	16	$40/46$
4	$2^4, 4^8$	16	$1^2, 8^3, 16^2$	8	$0/1$
4	$2^4, 4^8$	16	$2, 4^2, 8^2, 16^2$	4	$1/1$
4	$2^4, 4^8$	16	$2, 8^3, 16^2$	16	$2/2$
4	$2^4, 4^8$	16	$2, 8^3, 16^2$	8	$1/1$
4	$2^4, 4^8$	16	$2, 8^3, 16^2$	4	$1/1$
4	$2^4, 4^8$	16	$1^2, 8^3, 16^2$	16	$0/1$
4	$2^4, 4^8$	16	$1^2, 4^2, 8^2, 16^2$	8	$2/3$
4	$2^4, 4^8$	16	$1^2, 8^3, 16^2$	16	$0/1$
4	$2^4, 4^8$	16	$1^2, 8^3, 16^2$	8	$2/2$
4	$2^4, 4^8$	16	$1^2, 4^4, 8^3, 16$	16	$16/16$
4	$2^4, 4^8$	16	$1^2, 8^3, 16^2$	8	$0/1$
4	$2^4, 4^8$	8	$1^2, 2^6, 4^7, 8^2$	8	$152/162$
4	$2^4, 4^8$	8	$1^2, 2^2, 4^5, 8^4$	4	$4/4$

The table continues on the next page.

Table 1 – continuing from the previous page.

|A(SCG)|	O(SCG)	|A(BD)|	O(BD)	WA(SCG)	# preferred / #
4	$2^4, 4^8$	8	$1^2, 2^2, 4^5, 8^4$	8	8/8
4	$2^4, 4^8$	8	$2, 4^6, 8^4$	8	0/8
4	$2^4, 4^8$	8	$1^2, 4^6, 8^4$	8	16/36
4	$2^4, 4^8$	8	$1^2, 4^6, 8^4$	4	8/8
4	$2^4, 4^8$	4	$1^2, 2^{12}, 4^8$	4	112/125
4	4^{10}	24	$1^2, 2, 3^2, 4^3, 12, 24,$	24	0/1
4	4^{10}	12	$1^4, 3^2, 4^3, 12^3$	12	0/6
4	4^{10}	12	$1, 3^3, 12^4$	12	0/2
4	4^{10}	8	$1^4, 2^3, 4^4, 8^4$	8	56/71
4	4^{10}	8	$1^2, 2^2, 4^5, 8^4$	8	92/92
4	4^{10}	8	$2^5, 8^6$	8	12/14
4	4^{10}	8	$2^5, 8^6$	4	6/6
4	4^{10}	4	$1^2, 2^4, 4^{12}$	4	72/80
4	4^{10}	4	$1^{10}, 4^{12}$	4	24/45
2	2^{20}	8	$1^4, 2^3, 4^4, 8^4$	4	28/28
2	2^{20}	8	$1^2, 2^2, 4^5, 8^4$	4	46/46
2	2^{20}	8	$2^5, 8^6$	2	6/6
2	2^{20}	4	$1^4, 2^{11}, 4^8$	4	2328/2352
2	2^{20}	4	$1^4, 2^3, 4^{12}$	2	118/118
2	2^{20}	4	$1^4, 2^3, 4^{12}$	4	236/236
2	2^{20}	4	$1^2, 2^4, 4^{12}$	2	36/36
2	2^{20}	4	$1^{10}, 4^{12}$	2	12/12
2	2^{20}	2	$1^{10}, 2^{24}$	2	16496/16524

Table 2: Mixed Siamese color graphs for $q = 3$.

|A(SCG)|	O(SCG)	|A(BD)|	O(BD)	WA(SCG)	# preferred / #
36	$2^2, 18^2$	288	$1, 9, 24^2, 36$	144	2/2
36	$2^2, 18^2$	144	$1, 9, 24^2, 36$	72	0/2
36	$2^2, 18^2$	144	$1, 9, 12^4, 36$	144	2/2
36	$2^2, 18^2$	72	$1, 9, 12^4, 36$	36	0/1
18	$1^4, 9^4$	72	$1, 9, 12^4, 36$	36	0/3
12	$2^2, 6^2, 12^2$	48	$1, 3, 4, 6, 8, 12^2, 24^2$	24	4/4
12	$2^2, 6^2, 12^2$	24	$1, 2^2, 3, 4^2, 6^3, 12^3, 24$	24	10/10
12	$2^2, 6^2, 12^2$	24	$1, 2^2, 3, 4^2, 6^5, 12^4$	24	10/10
6	$2^2, 6^6$	24	$1, 3, 4^3, 6, 12^4, 24$	12	2/2
6	$2^2, 6^6$	24	$1, 3, 4^3, 6, 12^2, 24^2$	12	2/2
6	$2^2, 6^6$	12	$1, 2^6, 3^3, 6^8, 12^2$	12	2/2
6	$2^2, 6^6$	12	$1, 2^6, 3^3, 6^2, 12^5$	12	2/2
4	$2^4, 4^8$	32	$1^2, 4^3, 8^2, 16^4$	16	6/6
4	$2^4, 4^8$	16	$1^2, 2^2, 4^4, 8^7, 16$	8	4/4
4	$2^4, 4^8$	16	$1^2, 2^2, 4^4, 8^5, 16^2$	8	4/4
4	$2^4, 4^8$	16	$1^2, 4^7, 8^6, 16$	16	54/54
4	$2^4, 4^8$	16	$1^2, 4^5, 8^3, 16^3$	8	4/4
4	$2^4, 4^8$	8	$1^2, 2^8, 4^{11}, 8^4$	8	78/78
4	$2^4, 4^8$	8	$1^2, 2^6, 4^8, 8^6$	8	78/78
4	$2^4, 4^8$	8	$1^2, 2^4, 4^9, 8^6$	4	2/2
4	$2^4, 4^8$	8	$1^2, 2^2, 4^6, 8^8$	4	2/2
4	$2^4, 4^8$	8	$1^2, 2^4, 4^5, 8^8$	4	2/2
4	$2^4, 4^8$	8	$1^2, 2^2, 4^6, 8^8$	4	2/2
4	$2^4, 4^8$	8	$1^2, 4^7, 8^8$	8	0/2
4	$2^4, 4^8$	4	$1^2, 2^{14}, 4^{16}$	4	136/137
2	$1^8, 2^{16}$	4	$1^2, 2^8, 4^{19}$	4	0/3
2	2^{20}	8	$1^4, 2^3, 4^7, 8^7$	4	8/8
2	2^{20}	8	$1^2, 2^2, 4^8, 8^7$	4	44/44
2	2^{20}	4	$1^4, 2^{17}, 4^{14}$	4	530/530
2	2^{20}	4	$1^4, 2^{11}, 4^{17}$	4	530/530
2	2^{20}	4	$1^4, 2^9, 4^{18}$	2	22/22
2	2^{20}	4	$1^4, 2^3, 4^{21}$	2	22/22
2	2^{20}	4	$1^2, 2^4, 4^{21}$	2	36/36
2	2^{20}	2	$1^{10}, 2^{42}$	2	2880/2880