A note on parabolic and linear one-factorizations of the complete graph K_{p+1}

György Kiss[1,2], Gábor Korchmáros[3], Federico Romaniello[4] and Valentino Smaldore[5,*]

[1] Department of Geometry and HUN-REN-ELTE Geometric and Algebraic Combinatorics Research Group, Eötvös Loránd University, Pázmány s. 1/c, Budapest, 1117, Hungary

[2] FAMNIT, University of Primorska, Glagoljaška 8, Koper, 6000, Slovenia

[3] Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Viale dell'Ateneo Lucano 10, Potenza, 85100, Italy

[4] Dipartimento delle Culture Europee e del Mediterraneo, Università degli Studi della Basilicata, Via Lanera 20, Matera, 75100, Italy

[5] Dipartimento di Tecnica e Gestione dei Sistemi Industriali, Università degli Studi di Padova, Stradella S. Nicola 3, Vicenza, 361000, Italy

Abstract

One-factorizations of the complete graph K_n have wide applications, as an example they are often used for scheduling round-robin tournaments with n teams. In this note, we characterize parabolic and linear one-factorizations of complete graphs K_{p+1}, when p is an odd prime. This class of one-factorizations arises from the geometry of conics and lines in the affine plane $AG(2, p)$. We also include Magma computations for the cases $p \leq 19$.

Keywords

One-factorization, parabolas, affine lines

1. Introduction

Some of the most important sport competitions have a final classification which depends on a round-robin phase. In such tournaments, each participant plays at least once against any other team, and the final classification considers all results. Then tournaments as *Serie A* or *NBA* need efficient algorithms to compute all the possible match schedules, see [1, 2]. For example, some tournament uses *Berger's algorithm* [3] (developed by the chess player *Johann Berger*) which divides the n players into two equal sides, from 1 to $\frac{n}{2}$ and from $\frac{n}{2}+1$ to n; starts from the first pairing $\{1, n\}, \{2, n-1\}, \ldots, \left\{\frac{n}{2}, \frac{n}{2}+1\right\}$; and ends by giving some combinatorial argument to obtain all the other pairings. In our approach, we use one-factorizations of complete graphs. More precisely, a one-factorization of the complete graph K_n corresponds to a pairing in a round-robin tournament with n teams playing. In this note, we characterize parabolic and linear one-factorizations of complete graphs K_{p+1}, with p an odd prime number. In Section 2, we give preliminaries on graph theory and one-factorizations. In Sections 3 and 4, we give characterization results for parabolic and linear one-factorizations, where a one-factor is said to be *parabolic* or *linear* if it is represented by a parabola or a line, respectively. This note concludes with an Appendix containing computational results for the parabolic one-factorizations of K_{p+1}, $p = 13, 17, 19$, and the Magma code used by the authors.

2. Preliminaries

A graph $G = (V, E)$ is an incidence structure consisting of a set V of objects called vertices and a set E of object called edges. An edge $e \in E$ is denoted in the form $e = \{x, y\}$, where the vertices $x, y \in V$. Two vertices x and y connected by the edge $e = \{x, y\}$, are said to be adjacent. In what follows, neither multiple edges between the same pair of vertices nor loops, i.e. edges of the form $\{x, x\}$, are admitted. When every vertex has an equal number of edges incident to it, the graph is said to be regular. In particular, we will deal with a special class of regular graphs, in which each pair of vertices is connected by an edge. They are called complete graphs and are usually denoted by K_n, where n is the number of vertices. Let K_n be the complete graph with an even number n of vertices. A *one-factor* is a partition of its vertex set into $\frac{n}{2}$ disjoint edges. A *one-factorization* of K_n is a partition of its edge set into $n-1$ disjoint one-factors. For other notation or definitions on graphs not stated here, we refer the reader to [4]. Our approach to the problem of constructing one-factorizations of complete graphs is geometric, as in [5, 6, 7, 8, 9], and is based on techniques that have also been used for multigraphs, see [10, 11, 12, 13].

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*Corresponding author.

kissgy@cs.elte.hu (Gy. Kiss); gabor.korchmaros@unibas.it (G. Korchmáros); federico.romaniello@unibas.it (F. Romaniello); valentino.smaldore@unipd.it (V. Smaldore)

0000-0003-3312-9575 (Gy. Kiss); 0000-0002-2776-5754 (G. Korchmáros); 0000-0003-1166-3179 (F. Romaniello); 0000-0003-3433-3164 (V. Smaldore)

3. Parabolic one-factorizations

We adopt the same notation and terminology introduced in a paper by Korchmáros, Pace and Sonnino [5], and then used in subsequent paper by Kiss, Pace and Sonnino [6]. The reader may refer to [14] for notation and terminology on finite geometry not explicitly stated here. Let p be an odd prime. Fix a projective frame in $PG(2,p)$ with homogeneous coordinates $(X_0 : X_1 : X_2)$, and consider $PG(2,p)$ as $AG(2,p) \cup \ell_\infty$ where ℓ_∞ has equation $X_0 = 0$. As usual, the points of $AG(2,p)$ are written as (X,Y) with $X = \frac{X_1}{X_0}$ and $Y = \frac{X_2}{X_0}$. In $AG(2,p)$, let $\mathcal{P}_a$ be the parabola with affine equation $Y = X^2 + a$, where a varies in $\mathbb{Z}_p$, and $V_\infty = (0 : 0 : 1)$ the point at infinity of the line $X_1 = 0$. We remark that, in the projective closure of $AG(2,p)$, any two parabolas $\mathcal{P}_a$ and $\mathcal{P}_b$, with $a \neq b$, meet only at the point V_∞. Let P_i^k denote the affine point $(i + \frac{k}{2}, i^2 + ik)$ and P_i^∞ be the point $(0 : 1 : 2i)$ on the line at infinity ℓ_∞. The following result is easy to check:

Lemma 3.1. *For a fixed k, the points $P_0^k, P_1^k, \dots, P_{p-1}^k$ are on the parabola $\mathcal{P}_{-\frac{k^2}{4}}$.*

Definition 3.2. *Let $R = (u,v)$ be a point. The* symptome *of R is defined as $\mu_R = u^2 - v$.*

Let ℓ be a non-vertical line with equation $Y = mX + b$. The symptome *of ℓ is defined as $\lambda_\ell = m^2 + 4b$.*

We recall some basic properties of the parabolas $\mathcal{P}_a$. It is straightforward to check the following lemma:

Lemma 3.3. *Let $R = (u,v)$ be a point and $\ell\colon Y = mX + b$ be a non-vertical line. Then*

- *R is an external (internal) point of $\mathcal{P}_0$ if and only if μ_R is a square (non-square) element in* GF(p)*.*
- *ℓ is a tangent (secant or external line) to $\mathcal{P}_a$ if and only if $\lambda_\ell - 4a$ is 0 (a square or a non-square element in* GF(p)*).*
- *The pole of ℓ with respect to $\mathcal{P}_0$ is the point $L = (\frac{m}{2}, -b)$.*

The vertices of the complete graph K_{p+1} correspond to the points of $\mathcal{P}_0 \cup V_\infty$, while the edges of K_{p+1} correspond to the points of type P_i^k, with $k = 1, 2, \dots, \frac{p-1}{2}, \infty$. Thus the set of edges of K_{p+1} corresponds to the set of points

$$\mathcal{E} = \left(\bigcup_{k=1}^{\frac{p-1}{2}} \mathcal{P}_{-\frac{k^2}{4}} \right) \cup (\ell_\infty \setminus V_\infty).$$

These points are called *external* w.r.t. $\mathcal{P}_0$.

Definition 3.4. *A one-factor represented by a parabola $\mathcal{P}_a$ is a set of $\frac{p-1}{2}$ points of type P_j^k on $\mathcal{P}_a$, together with a suitable point at infinity. A one-factor so defined is referred to as a* parabolic one-factor.

Definition 3.5. *A* one-factor represented by a secant line *ℓ of $\mathcal{P}_0$ is a set consisting of $\frac{p-1}{2}$ points of $\mathcal{E}$ on ℓ, plus the pole of ℓ with respect to $\mathcal{P}_0$.*

A one-factor represented by an external line *ℓ of $\mathcal{P}_0$ is a set consisting of $\frac{p+1}{2}$ points of $\mathcal{E}$ on ℓ.*

Definition 3.6. *A one-factorization of K_{p+1} is called a* parabolic one-factorization *if $p-1$ of its one-factors are represented by parabolas and one of its one-factors is represented by a line.*

In [6] the authors proved the existence of an infinite family of parabolic one-factorization.

Theorem 3.7. *[6, Theorem 3.4] Let p be an odd prime. Then the complete graph K_{p+1} has a parabolic one-factorization.*

Proof. The proof is constructive. The set

$$F_0 = \left\{ P^k_{-\frac{k}{2}} : k = 1, 2, \dots, \frac{p-1}{2} \right\} \cup \{P_0^\infty\}$$

is a one-factor represented by the line secant line of $\mathcal{P}_0$ of equation $X = 0$, and P_0 is the pole. Now we define the sets

$$G_k = \left\{ P^k_{\frac{k}{2}+2jk} : j = 0, 1, \dots, \frac{p-3}{2} \right\} \cup \left\{ P^\infty_{-\frac{k}{2}} \right\},$$

$$H_k = \left\{ P^k_{\frac{k}{2}+(2j+1)k} : j = 0, 1, \dots, \frac{p-3}{2} \right\} \cup \left\{ P^\infty_{\frac{k}{2}} \right\}.$$

By Lemma 3.1 $G_k \setminus \left\{ P^\infty_{-\frac{k}{2}} \right\}$ and $H_K \setminus \left\{ P^\infty_{\frac{k}{2}} \right\}$ are disjoint subsets of the parabola $\mathcal{P}_{-\frac{k^2}{4}}$, and both G_k and H_k are one-factors represented by the parabola $\mathcal{P}_{-\frac{k^2}{4}}$. □

Definition 3.8. *A one-factorization of K_{p+1} is called an* almost parabolic one-factorization *if at least one of its one-factors is represented by $\mathcal{P}_{-\frac{k^2}{4}}$ for all $k \in \{1, 2, \dots \frac{p-1}{2}\}$, and all other of its one-factors are represented by lines.*

For $p < 11$ exhaustive computer search shows that each almost parabolic one-factorization of K_{p+1} is parabolic. In the rest of the paper we will assume $p \geq 11$.

Lemma 3.9. *The number of one-factors represented by lines in an almost parabolic one-factorization is either one, or at least $\lceil \frac{p+1}{4} \rceil$.*

Proof. If more than one one-factors are represented by lines, then there exist parabolas $\mathcal{P}_{-\frac{k^2}{4}}$ which represent only one one-factor. Hence $p - \frac{p-1}{2} = \frac{p+1}{2}$ of its points are covered by the lines represented the other one-factors. Any line meets $\mathcal{P}_{-\frac{k^2}{4}}$ in at most two points, so we need at least $\lceil \frac{p+1}{4} \rceil$ lines to cover these points. □

The following Lemma is a straightforward corollary of [6, Theorem 3.5].

Lemma 3.10. *Let $\mathcal{L}$ denote the set of points of type P_i^k belonging to the one-factors represented by lines in an almost parabolic one-factorization. Suppose that a one-factor is represented by the line $X = 0$. Then for each $k \in \{1, 2, \dots \frac{p-1}{2}\}$ either $\mathcal{L} \subset G_k$ or $\mathcal{L} \subset H_k$.*

Proposition 3.11. *An almost parabolic one-factorization contains at most two one-factors which are represented by vertical lines.*

Proof. Suppose to the contrary that it contains at least three one-factors which are represented by vertical lines. We may assume that the equations of the corresponding lines are $X = 0$, $X = u$, $X = v$ and $u - v = k' \in \{1, 2, \dots \frac{p-1}{2}\}$.

We also may assume that the line $X = v$ intersects $G_{k'}$. Then $v = \frac{k'}{2} + 2jk'$ where $j = 0, 1, \dots, \frac{p-3}{2}$. Hence

$$u = v + k' = \frac{k'}{2} + (2j + 1)k' \text{ where } j = 0, 1, \dots, \frac{p-3}{2}.$$

So the line $X = u$ intersects $H_{k'}$ contradicting Lemma 3.10. □

Lemma 3.12. *In* GF(p) *let T denote the set*

$$T = \Big\{(4j^2 + 4j)k^2 \colon j = 0, 1, \dots, \frac{p-1}{2}, \\ k = 1, 2, \dots, \frac{p-1}{2}\Big\}.$$

Then $|T| = p$.

Proof. First, we show that the cardinality of the set

$$U = \left\{4j^2 + 4j \colon j = 0, 1, \dots, \frac{p-1}{2}\right\}$$

is $\frac{p+1}{2}$. If $4j_1^2+4j_1 = 4j_2^2+4j_2$, then $(j_1-j_2)(j_1+j_2+1) = 0$. So for $j_1 \neq j_2$ we have $4j_1^2 + 4j_1 = 4j_2^2 + 4j_2$, because the sum in the second factor is not 0.

The set U obviously contains 0. Moreover, we claim that U contains both square and non-square elements. Suppose to the contrary, that all of the non-zero products $j(j+1)$ are either squares or non-squares. If 2 is a square, then $1 \cdot 2$ is a square, hence $2 \cdot 3$ is also a square which implies that 3 is a square. In the same way, step by step, we get that all of the elements $4, \dots, \frac{p-1}{2}, \frac{p+1}{2}$ are squares. Hence there are at least $\frac{p+1}{2}$ square elements, a contradiction. If 2 is a non-square, then $1 \cdot 2$ is a non-square, hence $2 \cdot 3$ is also a non-square, so 3 is a square. But 4 is also a square, hence $3 \cdot 4$ is a square, a contradiction again.

Now we show that the set

$$S = \left\{k^2 \colon k = 1, 2, \dots, \frac{p-1}{2}\right\}$$

equals to the set of square elements of GF(p). The cardinality of S is $\frac{p-1}{2}$, because $k_1^2 = k_2^2$ implies $(k_1 - k_2)(k_1 + k_2) = 0$, and the second factor is never 0, because $k_1 + k_2 \leq p - 1$.

Choose elements $u_1, u_2 \in U$ such that u_1 is a square and u_2 is a non-square. Then $u_1U \cap u_2U = \emptyset$, hence

$$|\{0\} \cup u_1U \cup u_2U| = 1 + \frac{p-1}{2} + \frac{p-1}{2} = p,$$

the statement is proved. □

Proposition 3.13. *If an almost parabolic one-factorization contains a one-factors which is represented by a vertical line, then it cannot contain one-factors represented by non-vertical lines.*

Proof. We may assume that the equation of the corresponding vertical line is $X = 0$. Suppose to the contrary that it contains the line $\ell \colon Y = mx + b$. We claim that there exists at least one $k \in \{1, 2, \dots, \frac{p-1}{2}\}$ such that $\ell \cap G_k \neq \emptyset \neq \ell \cap H_k$. By Lemma 3.12, there exist j and k such that $\lambda_\ell = (4j^2 + 4j)k^2$. Then

$$\lambda_\ell + k^2 = (4j^2 + 4j)k^2 + k^2 = ((2j + 1)k)^2,$$

so, by Lemma 3.3, ℓ intersects $\mathcal{P}_{-\frac{k^2}{4}}$ and the difference of the first coordinates of the two intersections is $(2j + 1)k$. Hence one of the two points belongs to G_k and the other one belongs to H_k. The statement follows from Lemma 3.10. □

The main result of this section is the following theorem, which derives from Propositions 3.9, 3.11 and 3.13.

Theorem 3.14. *Let p be an odd prime. If an almost parabolic one-factorization $\mathcal{F}$ of K_{p+1} contains a one-factor which is represented by a vertical line, then the one-factorization is parabolic.*

Proof. For $p < 11$ the statement follows from an exhaustive computer search, see [6].

If $p \geq 11$, then $\lceil \frac{p+1}{4} \rceil \geq 3$. Hence, by Lemma 3.9, $\mathcal{F}$ contains either one, or at least three one-factors that are represented by lines. In the former case, we are done. The latter case leads to a contradiction, since, by Lemma 3.11, the number of vertical lines is at most two, and by Lemma 3.13, there is no non-vertical line among the lines representing the one-factors. □

4. Linear one-factorizations

In this section, we consider one-factorizations whose all one-factors are represented by lines.

Let Ω be the set of points of an irreducible conic in $PG(2, q)$ with $q \geq 5$ odd.

Theorem 4.1. *Let K_{q+1} be a one-factorization on Ω whose one-factors are represented by lines. Then some of those lines are a chord of Ω.*

Proof. Let K_{q+1} be represented by the lines $\ell_1, \ldots, \ell_q$. Assume on the contrary that each those lines is an external line to Ω. Let $L_1, \ldots, L_q$ be the poles of $\ell_1, \ldots, \ell_q$ with respect to the orthogonal polarity π associated with Ω. For $i = 1, \ldots, q$, let φ_i be the involutory perspectivity with center L_i and axis ℓ_i which preserves Ω. Let $G \cong PGL(2, q)$ be the orthogonal group of Ω, i.e. the subgroup of $PGL(3, q)$ which commutes with π. Then G preserves Ω and acts on its points as $PGL(2, q)$ on the projective line over $\mathbb{F}_q$. Furthermore, $\varphi_i \in G$. Let $\mathcal{F}$ be the set consisting of φ_i, $i = 1, \ldots, q$ together with the identity of G. Since K_{q+1} is a one-factorization, for any two points $P, Q \in \Omega$ there exists a unique $\varphi \in \mathcal{F}$ such that $\varphi(P) = q$. Therefore, $\mathcal{F}$ is a sharply transitive permutation set on Ω containing the identity. From the classification of sharply transitive subsets of $PGL(2, q)$ [15], it turns out $\mathcal{F}$ is a subgroup of $PGL(2, q)$ of order $q + 1$. On the other hand, from Dickson's classification of subgroups of $PGL(2, q)$, the subgroups of $PGL(2, q)$ entirely consisting of involutions together with the identity, have order either 2 or 4. But then $q \leq 3$, a contradiction. □

We conclude by reporting a conjecture that is supported by computer-aided searches. With the aid of Magma [16] we verified that the conjecture holds for $p \leq 23$.

Conjecture 4.2. *[6, Conjecture 3.6] Let $p > 7$ be an odd prime, $\mathcal{F}$ be a one-factorization of the complete graph K_{p+1} such that each one-factor of $\mathcal{F}$ is either represented by a line or a parabola. Then $\mathcal{F}$ is either a parabolic one-factorization or each one-factor of $\mathcal{F}$ is represented by a line.*

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A. Tables

Corollary 3.14 states that almost parabolic one-factorization of K_{p+1}, p odd prime, containing a vertical line are parabolic. In this Appendix, we report examples of such parabolic one-factorizations of the complete graphs K_{14}, K_{18} and K_{20}, found by computations on Magma, [16]. K_4 admits only 1 one-factorization, and are well-known the 6 examples of 1-factorizations of K_6. We refer to [6] for the cases K_8 and K_{12}.

A.1. $p = 13$

$F_0 : \{P_{0,\infty} : (0:1:0), P_{1,6} : (1:0:3), P_{2,12} : (1:0:12),$
$P_{3,5} : (1:0:1), P_{4,11} : (1:0:9), P_{5,4} : (1:0:10),$
$P_{6,10} : (1:0:4)\}$

$G_1 : \{P_{6,\infty} : (0:1:12), P_{1,7} : (1:1:4), P_{1,9} : (1:3:12),$
$P_{1,11} : (1:5:2), P_{1,0} : (1:7:0), P_{1,2} : (1:9:6),$
$P_{1,4} : (1:11:7)\}$

$H_1 : \{P_{7,\infty} : (0:1:1), P_{1,8} : (1:2:7), P_{1,10} : (1:4:6),$
$P_{1,12} : (1:6:0), P_{1,1} : (1:8:2), P_{1,3} : (1:10:12),$
$P_{1,5} : (1:12:4)\}$

$G_2 : \{P_{12,\infty} : (0:1:11), P_{2,1} : (1:2:3), P_{2,5} : (1:6:9),$
$P_{2,9} : (1:10:8), P_{2,0} : (1:1:0), P_{2,4} : (1:5:11),$
$P_{2,8} : (1:9:2)\}$

$H_2 : \{P_{1,\infty} : (0:1:2), P_{2,3} : (1:4:2), P_{2,7} : (1:8:11),$
$P_{2,11} : (1:12:0), P_{2,2} : (1:3:8), P_{2,6} : (1:7:9),$
$P_{2,10} : (1:11:3)\}$

$G_3 : \{P_{5,\infty} : (0:1:10), P_{3,8} : (1:3:10), P_{3,1} : (1:9:4),$
$P_{3,7} : (1:2:5), P_{3,0} : (1:8:0), P_{3,6} : (1:1:2),$
$P_{3,12} : (1:7:11)\}$

$H_3 : \{P_{8,\infty} : (0:1:3), P_{3,11} : (1:6:11), P_{3,4} : (1:12:2),$
$P_{3,10} : (1:5:0), P_{3,3} : (1:11:5), P_{3,9} : (1:4:4),$
$P_{3,2} : (1:10:10)\}$

$G_4 : \{P_{11,\infty} : (0:1:9), P_{4,2} : (1:4:12), P_{4,10} : (1:12:10),$
$P_{4,5} : (1:7:6), P_{4,0} : (1:2:0), P_{4,8} : (1:10:5),$
$P_{4,3} : (1:5:8)\}$

$H_4 : \{P_{2,\infty} : (0:1:4), P_{4,6} : (1:8:8), P_{4,1} : (1:3:5),$
$P_{4,9} : (1:11:0), P_{4,4} : (1:6:6), P_{4,12} : (1:1:10),$
$P_{4,7} : (1:9:12)\}$

$G_5 : \{P_{4,\infty} : (0:1:8), P_{5,9} : (1:5:9), P_{5,6} : (1:2:1),$
$P_{5,3} : (1:12:11), P_{5,0} : (1:9:0), P_{5,10} : (1:6:7),$
$P_{5,7} : (1:3:6)\}$

$H_5 : \{P_{9,\infty} : (0:1:5), P_{5,1} : (1:10:6), P_{5,11} : (1:7:7),$
$P_{5,8} : (1:4:0), P_{5,5} : (1:1:11), P_{5,2} : (1:11:1),$
$P_{5,12} : (1:8:9)\}$

$G_6 : \{P_{10,\infty} : (0:1:7), P_{6,3} : (1:6:1), P_{6,2} : (1:5:3),$
$P_{6,1} : (1:4:7), P_{6,0} : (1:3:0), P_{6,12} : (1:2:8),$
$P_{6,11} : (1:1:5)\}$

$H_6 : \{P_{3,\infty} : (0:1:6), P_{6,9} : (1:12:5), P_{6,8} : (1:11:8),$
$P_{6,7} : (1:10:0), P_{6,6} : (1:9:7), P_{6,5} : (1:8:3),$
$P_{6,4} : (1:7:1)\}$

A.2. $p = 17$

$F_0 : \{P_{0,\infty} : (0:1:0), P_{1,8} : (1:0:4), P_{2,16} : (1:0:16),$
$P_{3,7} : (1:0:2), P_{4,15} : (1:0:13), P_{5,6} : (1:0:15),$
$P_{6,14} : (1:0:8), P_{7,5} : (1:0:9), P_{8,13} : (1:0:1)\}$

$G_1 : \{P_{8,\infty} : (0:1:16), P_{1,9} : (1:1:5), P_{1,11} : (1:3:13),$
$P_{1,13} : (1:5:12), P_{1,15} : (1:7:2), P_{1,0} : (1:9:0),$
$P_{1,2} : (1:11:6), P_{1,4} : (1:13:3), P_{1,6} : (1:15:8)\}$

$H_1 : \{P_{9,\infty} : (0:1:1), P_{1,10} : (1:2:8), P_{1,12} : (1:4:3),$
$P_{1,14} : (1:6:6), P_{1,16} : (1:8:0), P_{1,1} : (1:10:2),$
$P_{1,3} : (1:12:12), P_{1,5} : (1:14:13), P_{1,7} : (1:16:5)\}$

$G_2 : \{P_{16,\infty} : (0:1:15), P_{2,1} : (1:2:3), P_{2,5} : (1:6:1),$
$P_{2,9} : (1:10:14), P_{2,13} : (1:14:8), P_{2,0} : (1:1:0),$
$P_{2,4} : (1:5:7), P_{2,8} : (1:9:12), P_{2,12} : (1:13:15)\}$

$H_2 : \{P_{1,\infty} : (0:1:2), P_{2,3} : (1:4:15), P_{2,7} : (1:8:12),$
$P_{2,11} : (1:12:7), P_{2,15} : (1:16:0), P_{2,2} : (1:3:8),$
$P_{2,6} : (1:7:14), P_{2,10} : (1:11:1), P_{2,14} : (1:15:3)\}$

$G_3 : \{P_{7,\infty} : (0:1:14), P_{3,10} : (1:3:11), P_{3,16} : (1:9:15),$
$P_{3,5} : (1:15:6), P_{3,11} : (1:4:1), P_{3,0} : (1:10:0),$
$P_{3,6} : (1:16:3), P_{3,12} : (1:5:10), P_{3,1} : (1:11:4)\}$

$H_3 : \{P_{10,\infty} : (0:1:3), P_{3,13} : (1:6:4), P_{3,2} : (1:12:10),$
$P_{3,8} : (1:1:3), P_{3,14} : (1:7:0), P_{3,3} : (1:13:1),$
$P_{3,9} : (1:2:6), P_{3,15} : (1:8:15), P_{3,4} : (1:14:11)\}$

$G_4 : \{P_{15,\infty} : (0:1:13), P_{4,2} : (1:4:12), P_{4,10} : (1:12:4),$
$P_{4,1} : (1:3:5), P_{4,9} : (1:11:15), P_{4,0} : (1:2:0),$
$P_{4,8} : (1:10:11), P_{4,16} : (1:1:14), P_{4,7} : (1:9:9)\}$

$H_4 : \{P_{2,\infty} : (0:1:4), P_{4,6} : (1:8:9), P_{4,14} : (1:16:14),$
$P_{4,5} : (1:7:11), P_{4,13} : (1:15:0), P_{4,4} : (1:6:15),$
$P_{4,12} : (1:14:5), P_{4,3} : (1:5:4), P_{4,11} : (1:13:12)\}$

$G_5 : \{P_{6,\infty} : (0:1:12), P_{5,11} : (1:5:6), P_{5,4} : (1:15:2),$
$P_{5,14} : (1:8:11), P_{5,7} : (1:1:16), P_{5,0} : (1:11:0),$
$P_{5,10} : (1:4:14), P_{5,3} : (1:14:7), P_{5,13} : (1:7:13)\}$

$H_5 : \{P_{11,\infty} : (0:1:5), P_{5,16} : (1:10:13), P_{5,9} : (1:3:7),$
$P_{5,2} : (1:13:14), P_{5,12} : (1:6:0), P_{5,5} : (1:16:16),$
$P_{5,15} : (1:9:11), P_{5,8} : (1:2:2), P_{5,1} : (1:12:6)\}$

$G_6 : \{P_{14,\infty} : (0:1:11), P_{6,3} : (1:6:10), P_{6,15} : (1:1:9),$
$P_{6,10} : (1:13:7), P_{6,5} : (1:8:4), P_{6,0} : (1:3:0),$
$P_{6,12} : (1:15:12), P_{6,7} : (1:10:6), P_{6,2} : (1:5:16)\}$

$H_6 : \{P_{3,\infty} : (0:1:6), P_{6,9} : (1:12:16), P_{6,4} : (1:7:6),$
$P_{6,16} : (1:2:12), P_{6,11} : (1:14:0), P_{6,6} : (1:9:4),$
$P_{6,1} : (1:4:7), P_{6,13} : (1:16:9), P_{6,8} : (1:11:10)\}$

$G_7 : \{P_{5,\infty} : (0:1:1PZ0), P_{7,12} : (1:7:7), P_{7,9} : (1:4:8),$
$P_{7,6} : (1:1:10), P_{7,3} : (1:15:13), P_{7,0} : (1:12:0),$
$P_{7,14} : (1:9:5), P_{7,11} : (1:6:11), P_{7,8} : (1:3:1)\}$

$H_7 : \{P_{12,\infty} : (0:1:7), P_{7,2} : (1:14:1), P_{7,16} : (1:11:11),$
$P_{7,13} : (1:8:5), P_{7,10} : (1:5:0), P_{7,7} : (1:2:13),$
$P_{7,4} : (1:16:10), P_{7,1} : (1:13:8), P_{7,15} : (1:10:7)\}$

$G_8 : \{P_{13,\infty} : (0:1:9), P_{8,4} : (1:8:14), P_{8,3} : (1:7:16),$
$P_{8,2} : (1:6:3), P_{8,1} : (1:5:9), P_{8,0} : (1:4:0),$
$P_{8,16} : (1:3:10), P_{8,15} : (1:2:5), P_{8,14} : (1:1:2)\}$

$H_8 : \{P_{4,\infty} : (0:1:8), P_{8,12} : (1:16:2), P_{8,11} : (1:15:5),$
$P_{8,10} : (1:14:10), P_{8,9} : (1:13:0), P_{8,8} : (1:12:9),$
$P_{8,7} : (1:11:3), P_{8,6} : (1:10:16), P_{8,5} : (1:9:14)\}$

A.3. $p = 19$

$F_0 : \{P_{0,\infty} : (0:1:0), P_{1,9} : (1:0:14), P_{2,18} : (1:0:18),$
$P_{3,8} : (1:0:12), P_{4,17} : (1:0:15), P_{5,7} : (1:0:8),$
$P_{6,16} : (1:0:10), P_{7,6} : (1:0:2), P_{8,15} : (1:0:3),$
$P_{9,5} : (1:0:13)\}$

$G_1 : \{P_{9,\infty} : (0:1:18), P_{1,10} : (1:1:15), P_{1,12} : (1:3:4),$
$P_{1,14} : (1:5:1), P_{1,16} : (1:7:6), P_{1,18} : (1:9:0),$
$P_{1,1} : (1:11:2), P_{1,3} : (1:13:12), P_{1,5} : (1:15:11),$
$P_{1,7} : (1:17:18)\}$

$H_1 : \{P_{10,\infty} : (0:1:1), P_{1,11} : (1:2:18), P_{1,13} : (1:4:11),$
$P_{1,15} : (1:6:12), P_{1,17} : (1:8:2), P_{1,0} : (1:10:0),$
$P_{1,2} : (1:12:6), P_{1,4} : (1:14:1), P_{1,6} : (1:16:4),$
$P_{1,8} : (1:18:15)\}$

$G_2 : \{P_{18,\infty} : (0:1:17), P_{2,1} : (1:2:3), P_{2,5} : (1:6:16),$
$P_{2,9} : (1:10:4), P_{2,13} : (1:14:5), P_{2,17} : (1:18:0),$
$P_{2,2} : (1:3:8), P_{2,6} : (1:7:10), P_{2,10} : (1:11:6),$
$P_{2,14} : (1:15:15)\}$

$H_2 : \{P_{1,\infty} : (0:1:2), P_{2,3} : (1:4:15), P_{2,7} : (1:8:6),$
$P_{2,11} : (1:12:10), P_{2,15} : (1:16:8), P_{2,0} : (1:1:0),$
$P_{2,4} : (1:5:5), P_{2,8} : (1:9:4), P_{2,12} : (1:13:16),$
$P_{2,16} : (1:17:3)\}$

$G_3 : \{P_{8,\infty} : (0:1:16), P_{3,11} : (1:3:2), P_{3,17} : (1:9:17),$
$P_{3,4} : (1:15:9), P_{3,10} : (1:2:16), P_{3,16} : (1:8:0),$
$P_{3,3} : (1:14:18), P_{3,9} : (1:1:13), P_{3,15} : (1:7:4),$
$P_{3,2} : (1:13:10)\}$

$H_3 : \{P_{11,\infty} : (0:1:3), P_{3,14} : (1:6:10), P_{3,1} : (1:12:4),$
$P_{3,7} : (1:18:13), P_{3,13} : (1:5:18), P_{3,0} : (1:11:0),$
$P_{3,6} : (1:17:16), P_{3,12} : (1:4:9), P_{3,18} : (1:10:17),$
$P_{3,5} : (1:16:2)\}$

$G_4 : \{P_{17,\infty} : (0:1:15), P_{4,2} : (1:4:12), P_{4,10} : (1:12:7),$
$P_{4,18} : (1:1:16), P_{4,7} : (1:9:1), P_{4,15} : (1:17:0),$
$P_{4,4} : (1:6:13), P_{4,12} : (1:14:2), P_{4,1} : (1:3:5),$
$P_{4,9} : (1:11:3)\}$

$H_4 : \{P_{2,\infty} : (0:1:4), P_{4,6} : (1:8:3), P_{4,14} : (1:16:5),$
$P_{4,3} : (1:5:2), P_{4,11} : (1:13:13), P_{4,0} : (1:2:0),$
$P_{4,8} : (1:10:1), P_{4,16} : (1:18:16), P_{4,5} : (1:7:7),$
$P_{4,13} : (1:15:12)\}$

$G_5 : \{P_{7,\infty} : (0:1:14), P_{5,12} : (1:5:14), P_{5,3} : (1:15:5),$
$P_{5,13} : (1:6:6), P_{5,4} : (1:16:17), P_{5,14} : (1:7:0),$
$P_{5,5} : (1:17:12), P_{5,15} : (1:8:15), P_{5,6} : (1:18:9),$
$P_{5,16} : (1:9:13)\}$

$H_5 : \{P_{12,\infty} : (0:1:5), P_{5,17} : (1:10:13), P_{5,8} : (1:1:9),$
$P_{5,18} : (1:11:15), P_{5,9} : (1:2:12), P_{5,0} : (1:12:0),$
$P_{5,10} : (1:3:17), P_{5,1} : (1:13:6), P_{5,11} : (1:4:5),$
$P_{5,2} : (1:14:14)\}$

$G_6 : \{P_{16,\infty} : (0:1:13), P_{6,3} : (1:6:8), P_{6,15} : (1:18:11),$
$P_{6,8} : (1:11:17), P_{6,1} : (1:4:7), P_{6,13} : (1:16:0),$
$P_{6,6} : (1:9:15), P_{6,18} : (1:2:14), P_{6,11} : (1:14:16),$
$P_{6,4} : (1:7:2)\}$

$H_6 : \{P_{3,\infty} : (0:1:6), P_{6,9} : (1:12:2), P_{6,2} : (1:5:16),$
$P_{6,14} : (1:17:14), P_{6,7} : (1:10:15), P_{6,0} : (1:3:0),$
$P_{6,12} : (1:15:7), P_{6,5} : (1:8:17), P_{6,17} : (1:1:11),$
$P_{6,10} : (1:13:8)\}$

$G_7 : \{P_{6,\infty} : (0:1:12), P_{7,13} : (1:7:13), P_{7,8} : (1:2:6),$
$P_{7,3} : (1:16:11), P_{7,17} : (1:11:9), P_{7,12} : (1:6:0),$
$P_{7,7} : (1:1:3), P_{7,2} : (1:15:18), P_{7,16} : (1:10:7),$
$P_{7,11} : (1:5:8)\}$

$H_7 : \{P_{13,\infty} : (0:1:7), P_{7,1} : (1:14:8), P_{7,15} : (1:9:7),$
$P_{7,10} : (1:4:18), P_{7,5} : (1:18:3), P_{7,0} : (1:13:0),$
$P_{7,14} : (1:8:9), P_{7,9} : (1:3:11), P_{7,4} : (1:17:6),$
$P_{7,18} : (1:12:13)\}$

$G_8 : \{P_{15,\infty} : (0:1:11), P_{8,4} : (1:8:10), P_{8,1} : (1:5:9),$
$P_{8,17} : (1:2:7), P_{8,14} : (1:18:4), P_{8,11} : (1:15:0),$
$P_{8,8} : (1:12:14), P_{8,5} : (1:9:8), P_{8,2} : (1:6:1),$
$P_{8,18} : (1:3:12)\}$

$H_8 : \{P_{4,\infty} : (0:1:8), P_{8,12} : (1:16:12), P_{8,9} : (1:13:1),$
$P_{8,6} : (1:10:8), P_{8,3} : (1:7:14), P_{8,0} : (1:4:0),$
$P_{8,16} : (1:1:4), P_{8,13} : (1:17:7), P_{8,10} : (1:14:9),$
$P_{8,7} : (1:11:10)\}$

$G_9 : \{P_{5,\infty} : (0:1:10), P_{9,14} : (1:9:18), P_{9,13} : (1:8:1),$
$P_{9,12} : (1:7:5), P_{9,11} : (1:6:11), P_{9,10} : (1:5:0),$
$P_{9,9} : (1:4:10), P_{9,8} : (1:3:3), P_{9,7} : (1:2:17),$
$P_{9,6} : (1:1:14)\}$

$H_9 : \{P_{14,\infty} : (0:1:9), P_{9,4} : (1:18:14), P_{9,3} : (1:17:17),$
$P_{9,2} : (1:16:3), P_{9,1} : (1:15:10), P_{9,0} : (1:14:0),$
$P_{9,18} : (1:13:11), P_{9,17} : (1:12:5), P_{9,16} : (1:11:1),$
$P_{9,15} : (1:10:18)\}$

B. Magma code

Below, we report the code used to find the parabolic one-factorizations.

```
p:=19;
F:=GF(p);
AG:=AffineSpace(F,3);
r:=(p-1)/2;
rr:=(p-3)/2;
F0:={AG![0,1,0]};
for k in [1..r] do
kk:=F!k;
l:=AG![1,0,(-kk/2)^2+(-kk/2)*kk];
F0:=F0 join {l};
end for;
printf "F 0 : %o\n" ,F0;
```

```
for k in [1..r] do
kk:=F!k;
G:={AG![0,1,-kk]};
H:={AG![0,1,kk]};
for j in [0..rr] do
jj:=F!j;
i:=F!(kk/2+2*jj*kk);
ii:=F!(kk/2+(2*jj+1)*kk);
m:=AG![1,i+kk/2,i^2+i*kk];
n:=AG![1,ii+kk/2,ii^2+ii*kk];
G:=G join {m};
H:=H join {n};
end for;
printf "G %o : %o\n" ,kk,G;
printf "H %o : %o\n" ,kk,H;
end for;
```