Large convex independent subset in sum of two point sets

Ondřej Bílka
Department of Applied Mathematics
Charles University
Malostranské Nám. 25
118 00 Praha 1, Czech Republic
neleai@seznam.cz

1 Abstract

We answer question of Eisenbrand, Pach, Rothvoss and Sopher if the sum of two finite point sets in the plane can have superlinear convex independent set. We show a construction which gives us convex independent sets with size which is asymptotically tight.

2 Introduction

Halman et al. [2] studied maximal number $E(n)$ of edges between n points in the plane such that their midpoints form a convex independent set. They asked if $E(n)$ is linear or quadratic.

Motivated by this question Eisenbrand et al. [1] studied a more general question: What is the maximal size $M(m, n)$ of convex independent subset of $P + Q, |P| = m, |Q| = n$?

This directly relates to previous question because we can write set of midpoints of P as $1/2(P + P)$. If we have convex independent set in $P + Q$ then midpoints of $2P(\cup Q)$ contain this set too. Eisenbrand at al. showed the following upper bound on the maximum size of convex independent set:

$$M(m, n) = O(m^{2/3}n^{2/3} + m + n).$$

They mentioned they don't know any superlinear lower bound of M(m,n). We will prove that when $m = n$ their bound $M(n, n) = O(n^{4/3})$ is tight.

Theorem 1. *For every $n = m^2$ (where m is an integer) there exist sets J and K of size $|J| = |K| = n$ such that the sum $J + K$ contains a convex independent subset of size $\Omega(n^{4/3})$.*

3 Definitions

By the *sum* $A + B$ of planar point sets A and B we mean the set $\{a + b | a \in A, b \in B\}$.
By direction $dir(v)$ of a vector $v = (x, y), x \neq 0$, we mean the ratio $dir(v) = y/x$.
Let G be the square grid $G = \{(i, j) | i, j \in \{1, 2, \ldots, m\}\}$.
We call $L \subset G$ a *line in* G if there exists a line l such that $L = l \cap G$. We restrict ourselves to lines in G with direction $0 < dir(l) < 1$.

We call a set $U \subset R^2$ a *cup* if U is a subset of the graph of a convex function.
Let z be the mapping $z(x) = \epsilon 3^x$, where ϵ is choosen to satisfy $z(m^2+m) < 1/m^2$.
Let $\phi : R^2 \to R^2$ be mapping $\phi(i,j) = (i, j + z(mi+j))$.
Observe ϕ is direction preserving in this sense:
(1) If $a, b, c, d \in G$ and $0 < dir(b-a) < dir(d-c) < 1$ then $dir(\phi(b)-\phi(a)) < dir(\phi(d)-\phi(c))$.

4 Proof of Theorem 1

Proof. We put $K := \phi(G)$ and describe J implicitly. The sum $J+K$ consists of n shifted copies of K. Take the set S of n lines of G with largest sizes. For any direction a/b consider the set $S_{a/b}$ of lines from S with direction a/b.

Claim 1. *For any direction $a/b \in (0,1)$ there exists a mapping $t : S_{a/b} \to R^2$ such that the set $U_{a/b} = \bigcup_{L \in S_{a/b}} \phi(L) + t(L)$ is a cup. Moreover no two points of $U_{a/b}$ coincide.*

Proof. First we define linear mapping $f : R^2 -> R^2$ such that

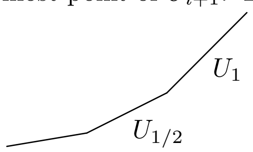
$$f(x,y) = \left(a\frac{mx+y}{ma+b}, b\frac{mx+y}{ma+b} \right).$$

For any line $L \in S_{a/b}$, $f(L)$ is a translation of L which follows from $f(a,b) = (a,b)$.
Now observe that for any $x, y \in R$

$$\phi(x,y) - (x,y) = (0, z(mx+y)) = \phi(f(x,y)) - f(x,y).$$

Thus, $\phi(f(L))$ is the translation of $\phi(L)$ we seek. The image of ϕf coincides with the graph of the convex function $y = b/ax + \epsilon 3^{(m+b/a)x}$. Therefore $U_{a/b}$ is a cup. No two points of $U_{a/b}$ coincide because $mx+y$ restricted to G is an injective mapping. □

Now let $U_1, U_2, \ldots, U_k$ be the sets $U_{a/b}$ sorted by increasing direction. We shift these sets in such a way that the rightmost point of U_i coincides with the leftmost point of U_{i+1}. Then the set $U_1 \cup \ldots \cup U_k$ is a cup due to (1).



Now we need to estimate the size of this set. For direction a/b the set $U_{a/b}$ has $\frac{bm}{2}$ lines starting at $(i,j), 0 \le i < n/2, 0 \le j < b$. Each of these lines has at least $\frac{m}{2b}$ points. Since the number of divisors of b is at most $b/2$ the number of directions with given b is at least $\frac{b}{2}$. The number of lines we use is at most $\sum_{b=1}^{k} 1/2 \sum_{a=0}^{b} |U_{a/b}| = 1/2 \sum_{b=1}^{k} \frac{b}{2}\frac{bm}{2}$. Set $k = m^{1/3}$ to ensure we use less than $n = m^2$ lines. The number of points on these lines is at least $\sum_{b=1}^{k} \frac{m}{2b} b/2 \frac{bm}{2} = \sum_{b=1}^{m^{1/3}} bm^2/8 = \Omega(m^{2/3} m^2) = \Omega(n^{4/3})$.

□

References

[1] F. Eisenbrand, J. Pach, T. Rothvoss, and N. B. Sopher: *Convexly independent subsets of the Minkovski sum of planar point sets*, Electronic J Comb. **15** (2008) #N8.

[2] N. Halman, S. Onn, and U. G. Rohtblum: *The convex dimension of a graph* Discrete Applied Math.**155** (2007), 1373-1383.