Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

SVOČ 2009

Soutěž vysokoškoláků ve vědecké odborné činnosti
v matematice a informatice

Jakub Bulín

CSP Dichotomy for Special Polyads

(Matematické struktury – Algebra, topologie a geometrie)

CSP DICHOTOMY FOR SPECIAL POLYADS

JAKUB BULÍN

ABSTRACT. For a digraph $\mathbb{G}$, the Constraint Satisfaction Problem with template $\mathbb{G}$, or CSP($\mathbb{G}$), is the problem of deciding whether a given input digraph $\mathbb{H}$ admits a homomorphism to $\mathbb{G}$. The CSP dichotomy conjecture of Feder and Vardi states that for any digraph $\mathbb{G}$, CSP($\mathbb{G}$) is either in P or NP-complete. Barto, Kozik, Maróti and Niven [3] confirmed the conjecture for a class of oriented trees called special triads. We generalize this result, establishing the dichotomy for a class of oriented trees which we call special polyads.

1. Introduction

Let $\mathbb{G}$ be a fixed finite digraph. The Constraint Satisfaction Problem with template $\mathbb{G}$, or CSP($\mathbb{G}$) for short, is the following decision problem:

INPUT: A finite digraph $\mathbb{H}$.
QUESTION: Is there a homomorphism from $\mathbb{H}$ to $\mathbb{G}$?

In graph theory, CSP($\mathbb{G}$) is also called $\mathbb{G}$-coloring problem. This class of problems has recently recieved a lot of attention, mainly because of the work of Feder and Vardi [7] from 1999. In this article the authors conjectured a large natural class of NP decision problems avoiding the complexity classes between P and NP-complete (assuming that P$\neq$NP). Many natural decision problems, such as k-SAT, graph k-colorability or solving systems of linear equations over finite fields belong to this class. In the same article they proved that each such problem can be expressed as CSP($\mathbb{G}$) for some digraph $\mathbb{G}$. Therefore their dichotomy conjecture can be formulated as follows:

The CSP dichotomy conjecture. *For every digraph $\mathbb{G}$,* CSP($\mathbb{G}$) *is either tractable or NP-complete.*

We say for brevity that $\mathbb{G}$ is tractable (NP-complete) if CSP($\mathbb{G}$) is tractable (NP-complete).

The dichotomy was established for a number of special cases, including oriented paths (which are all tractable) [8], oriented cycles [6], undirected graphs [9] and many others. The work of Jeavons, Cohen and Gyssens [10], refined by Bulatov, Jeavons and Krokhin [4], has shown a strong connection between the constraint satisfaction problem and universal algebra. This "algebraic approach" led to a rapid development of the subject and is essential to our paper. For more information on the algebraic approach to CSP, see

Date: April 30, 2009.
Key words and phrases. Constraint satisfaction problem, graph coloring, bounded width, special triad.

the survey of Krokhin, Bulatov and Jeavons [11]. Using the algebraic approach (in particular, a result of Maróti and McKenzie [13]), Barto, Kozik and Niven [2] established the CSP dichotomy for digraphs without sources or sinks (i.e., digraphs such that each vertex has an incoming and an outgoing edge).

In the class of all digraphs, oriented trees are in some sense very "far" from digraphs without sources or sinks. Except the oriented paths, the simplest class of oriented trees are the *triads* (i.e., oriented trees with one vertex of degree 3 and all other vertices of degree 1 or 2). Though the dichotomy conjecture for triads remains open, it was confirmed by Barto, Kozik, Maróti and Niven [3] for the so-called *special triads*, a certain class of triads possessing enough structure to provide a structural description of the tractable and NP-complete cases. Our paper generalizes their result to the *special polyads* (which will be defined later). A *polyad* is an oriented tree with one vertex of degree $n > 0$ and all other vertices of degree 1 or 2. Special polyads are a straightforward generalization of special triads.

A digraph $\mathbb{G}$ is said to have *bounded width* if CSP($\mathbb{G}$) can be solved by a certain polynomial-time algorithm called *Local Consistency Checking* (see [7]). It was proved earlier that if $\mathbb{G}$ has a compatible *majority operation* [7] or compatible *totally symmetric idempotent operations* of all arities [5], then it has bounded width (and thus CSP($\mathbb{G}$) is tractable). In [12], Larose and Zádori conjectured a full characterization of digraphs with bounded width. This conjecture was recently confirmed by Barto and Kozik [1]. Our paper relies on their result that digraphs with compatible *weak near-unanimity operations* of almost all arities have bounded width (see Theorem 3.4 below).

In [3], the authors proved that every special triad is either NP-complete or it has a compatible majority operation or compatible totally symmetric idempotent operations of all arities. We concentrated on the special polyads for several reasons. Though the special polyads do possess the same kind of structure as the special triads, allowing us to apply some of the techniques used in [3], it was not obvious whether the results from [3] can be extended to them.

We were also interested in the following question: Will every tractable special polyad be tractable for a "simple" reason, by which we mean satisfying some strong conditions ensuring tractability (e.g., possessing a compatible majority operation, near-unanimity operation or totally symmetric idempotent operations of all arities)? We were not able to find such a strong condition for every tractable special polyad, therefore we need the result from [1] in its full strength. Moreover, we wanted to determine whether there exist tractable special polyads without bounded width. The answer to this question is negative. We believe that the techniques developed in this article can be applied to a far broader class of oriented trees.

2. Preliminaries

A *digraph* $\mathbb{G}$ is a set of *vertices* $V(\mathbb{G})$ together with a set of *edges* $E(\mathbb{G}) \subseteq V(\mathbb{G})^2$. For $a, b \in V(\mathbb{G})$ such that $\langle a, b \rangle \in E(\mathbb{G})$, we write $a \overset{\mathbb{G}}{\to} b$. A *homomorphism* $f : \mathbb{H} \to \mathbb{G}$ is a mapping from $V(\mathbb{H})$ to $V(\mathbb{G})$ which preserve edges, i.e., for $a, b \in V(\mathbb{H})$ with $a \overset{\mathbb{H}}{\to} b$ we have that $f(a) \overset{\mathbb{G}}{\to} f(b)$. An *endomorphism* is a homomorphism from $\mathbb{G}$ to $\mathbb{G}$. A digraph $\mathbb{H}$ is called a *subgraph* of $\mathbb{G}$ if $V(\mathbb{H}) \subseteq V(\mathbb{G})$ and $E(\mathbb{H}) \subseteq E(\mathbb{G})$.

A digraph is a core if every its endomorphism is surjective. For a digraph $\mathbb{G}$, the *core of* $\mathbb{G}$ is the smallest subgraph $\mathbb{G}'$ of $\mathbb{G}$ such that there exists an endomorphism $f : \mathbb{G} \to \mathbb{G}'$ (it is determined uniquely up to isomorphism). It is easily seen that a digraph $\mathbb{H}$ maps homomorphically to $\mathbb{G}$ if and only if there exists a homomorphism $\mathbb{H} \to \mathbb{G}'$. In particular, CSP($\mathbb{G}$) has the same complexity as CSP($\mathbb{G}'$).

By an *oriented tree* we mean a digraph obtained from an undirected tree (i.e., an undirected graph without cycles) by orienting its edges. Let $\mathbb{G}$ be an oriented tree. To each $a \in V(\mathbb{G})$ we can uniquely assign a nonnegative integer *level*(a) (called the *level* of a) such that the following holds:

(i) If $a \overset{\mathbb{G}}{\to} b$, then $level(b) = level(a) + 1$.
(ii) There exists a vertex with level 0.

The *height* of $\mathbb{G}$, denoted by $\mathrm{hgt}(\mathbb{G})$, is the highest level of a vertex of $\mathbb{G}$.

An *oriented path* of length n is a digraph with vertices $v_0, v_1, \ldots, v_n$ and edges $e_0, e_1, \ldots, e_{n-1}$ such that e_i is either $\langle v_i, v_{i+1}\rangle$ or $\langle v_{i+1}, v_i\rangle$. By a *distance* of two vertices a, b in a digraph $\mathbb{G}$ we mean the minimal length of a directed path $\mathbb{P}$ such that $\mathbb{P}$ is a subgraph of $\mathbb{G}$ and $a, b \in V(\mathbb{P})$.

Let $\mathbb{P}$ be an oriented path. We define the *net length* of $\mathbb{P}$ to be the number

$$\mathrm{net}(\mathbb{P}) = |\#\{i : v_i \overset{\mathbb{P}}{\to} v_{i+1}\} - \#\{i : v_{i+1} \overset{\mathbb{P}}{\to} v_i\}|.$$

Note that the height of $\mathbb{P}$ is precisely $\mathrm{net}(\mathbb{P})$. $\mathbb{P}$ is called *minimal* if it has precisely one vertex of level 0 (the *initial* vertex) and it is one of the vertices v_0, v_n and precisely one vertex of level $\mathrm{net}(\mathbb{P})$ (the *terminal* vertex) which is the other one of v_0, v_n. Below is an example of a minimal path:

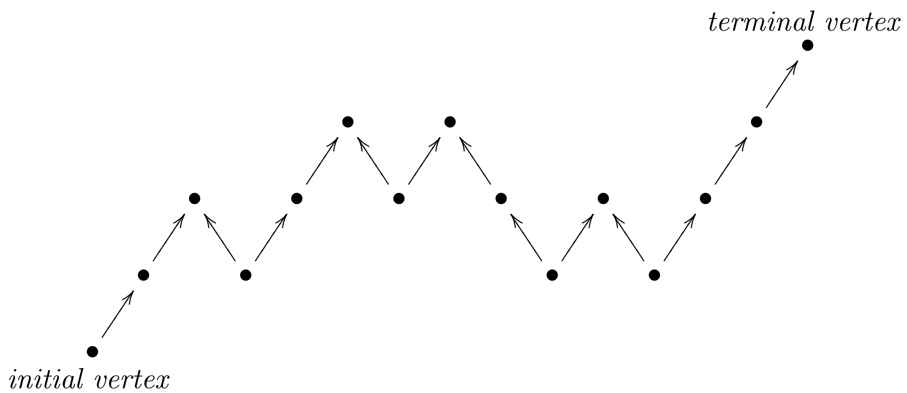

FIGURE 1. A minimal path of net length 4

3. CSP and Compatible weak-NUs

In this section we introduce the *weak near-unanimity* operations and their connection to the complexity of CSP($\mathbb{G}$). Recall that by an *r-ary operation* on a set A we wean a mapping $A^r \to A$, f is said to be *idempotent* if it satisfies $f(a, a, \ldots, a) = a$ for all $a \in A$.

Definition 3.1. Let $r \geq 3$. An r-ary operation ω on A is called a *weak near-unanimity* operation (or a *weak-NU*) if it is idempotent and satisfies

$$\omega(a, \dots, a, b) = \omega(a, \dots, a, b, a) = \cdots = \omega(b, a, \dots, a)$$

for all $a, b \in A$. We also define the binary operation $\circ_\omega$ by setting

$$a \circ_\omega b = \omega(a, \dots, a, b).$$

We will be interested in weak near-unanimity operations *compatible with* a given digraph:

Definition 3.2. Let $\mathbb{G}$ be a digraph and let ω be an r-ary operation on $V(\mathbb{G})$. We say that ω is *compatible with* $\mathbb{G}$ (or is a *compatible operation of* $\mathbb{G}$) if it satisfies the following condition: if $a_i, b_i \in V(\mathbb{G})$ and $a_i \overset{\mathbb{G}}{\to} b_i$ for $i = 1, \dots, r$, then $\omega(a_1, \dots, a_r) \overset{\mathbb{G}}{\to} \omega(b_1, \dots, b_r)$.

It can be easily seen that an operation obtained by composing operations compatible with $\mathbb{G}$ is also compatible with $\mathbb{G}$. In particular if ω is a weak-NU operation compatible with $\mathbb{G}$, then $\circ_\omega$ is also compatible with $\mathbb{G}$, as we can obtain it by composing ω with the *projection operations* (i.e. the operations $p_r^i(x_1, \dots, x_r) = x_i$, which are indeed compatible with $\mathbb{G}$).

In the rest of this section we introduce two theorems connecting the computational complexity of CSP($\mathbb{G}$) with existence of weak near-unanimity operations compatible with $\mathbb{G}$. We will later use these algebraic tools to prove tractability or NP-completeness of the "special polyads" defined in the next section. The following theorem is a combination of a result of Bulatov, Jeavons and Krokhin from [4] and a result of Maróti and McKenzie [13].

Theorem 3.3. *Let $\mathbb{G}$ be a digraph. If the core of $\mathbb{G}$ admits no compatible weak-NU operation, then* CSP($\mathbb{G}$) *is NP-complete.*

The next theorem is a recent result of Barto and Kozik [1].

Theorem 3.4. *Let $\mathbb{G}$ be a digraph. If the core of $\mathbb{G}$ admits compatible weak-NU operations of almost all arities (i.e., there exists k_0 such that for all $k \geq k_0$ the core of $\mathbb{G}$ admits a compatible k-ary weak-NU), then $\mathbb{G}$ has bounded width.*

4. Special polyads, Main theorem

In this section we define the *special polyads*, a certain class of oriented trees generalizing the special triads treated in [3], as well as their cores. An *n-ad* is an oriented tree which has precisely one vertex of degree n and all other vertices of degree 1 or 2.

Definition 4.1. Let n and k be nonnegative integers, $n \geq 1$ and $0 \leq k \leq n$. Let $\mathbb{P}_1$, $\mathbb{P}_2$, …, $\mathbb{P}_n$, $\mathbb{P}'_1$, $\mathbb{P}'_2$, …, $\mathbb{P}'_n$ be minimal oriented paths of the same net length. For each $i \in \{1, 2, \dots, n\}$, let the initial and terminal vertices of $\mathbb{P}'_i$ be i and $\widehat{i}$, respectively.

A *special n-ad* given by the paths $\mathbb{P}_1$, $\mathbb{P}_2$, …, $\mathbb{P}_n$, $\mathbb{P}'_1$, $\mathbb{P}'_2$, …, $\mathbb{P}'_n$ is the oriented tree obtained by identifying the terminal vertices of $\mathbb{P}_i$ with $\widehat{i}$ for $i = 1, 2, \dots, n$ and identifying the initial vertices of $\mathbb{P}_1$, $\mathbb{P}_2$, …, $\mathbb{P}_n$ to a single vertex 0 (see the figure below, arrows indicate "direction" of paths).

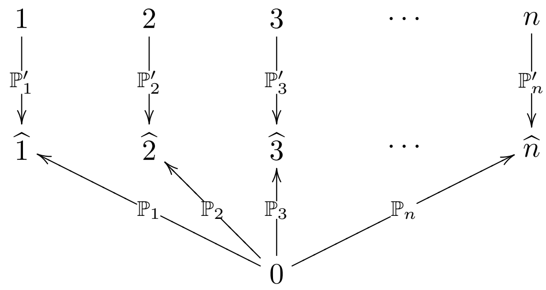

FIGURE 2. A special n-ad

The core of a special n-ad may not be a special n'-ad for any $n' \leq n$. In order to be able to use Theorem 3.3, we introduce the following notion:

Definition 4.2. Let $n \geq 1$ and $0 \leq k \leq n$. By a *special n-ad with k half-branches* given by the paths $\mathbb{P}_1$, $\mathbb{P}_2$, ..., $\mathbb{P}_n$, $\mathbb{P}'_1$, $\mathbb{P}'_2$, ..., $\mathbb{P}'_{n-k}$ we mean the oriented tree obtained from a special n-ad given by $\mathbb{P}_1$, $\mathbb{P}_2$, ..., $\mathbb{P}_n$, $\mathbb{P}'_1$, $\mathbb{P}'_2$, ..., $\mathbb{P}'_n$ (for arbitrary $\mathbb{P}'_{n-k+1}$, $\mathbb{P}'_{n-k+2}$, ..., $\mathbb{P}'_n$) by removing all the vertices of $\mathbb{P}'_{n-k+i}$ except of $\widehat{i}$ for each $i = 1, \dots, k$.

We denote the set of vertices from $V(\mathbb{G})$ of level 0 by Low and the set of vertices of maximal level by Upp, i. e. Low $= \{0, 1, 2, \dots, n-k\}$ and Upp $= \{\widehat{1}, \widehat{2}, \dots, \widehat{n}\}$. We put Half $= \{\widehat{n-k+1}, \widehat{n-k+2}, \dots, \widehat{n}\}$. Let us also define Paths($\mathbb{G}$) to be the set $\{\mathbb{P}_1, \mathbb{P}_2, \dots, \mathbb{P}_n, \mathbb{P}'_1, \mathbb{P}'_2, \dots, \mathbb{P}'_{n-k}\}$.

By a *special polyad* we will mean a special n-ad with k half-branches with n and k arbitrary.

Below is an example of a special polyad, namely special 5-ad with 2 half-branches:

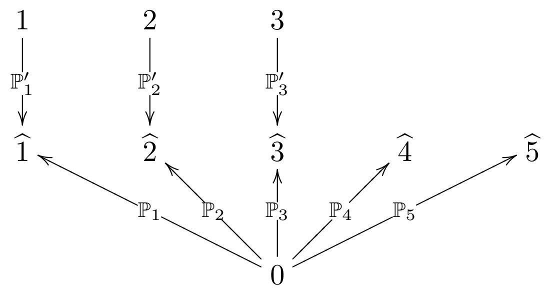

FIGURE 3. A special 5-ad with 2 half-branches

Note that in our terminology, a special triad from [3] is a special 3-ad with 0 half-branches. Now we can state the following observation:

Lemma 4.3. *Let $\mathbb{G}$ be a special polyad. Then the core of $\mathbb{G}$ is also a special polyad.*

Proof. It is easily seen that a homomorphism from a minimal path of net length l to an oriented tree of height l maps the initial vertex to a vertex of level 0 and the terminal vertex to a vertex of level l. The rest follows directly from this fact. $\square$

The following theorem is the main result of this paper:

Theorem 4.4. *For every special polyad $\mathbb{G}$, either* $\mathrm{CSP}(\mathbb{G})$ *is NP-complete or $\mathbb{G}$ has bounded width (and thus is tractable).*

We will prove Theorem 4.4 in section 6.

5. $\mathrm{Comp}(\mathbb{G})$ and nice weak-NUs

Let $\mathbb{G}$ be a special polyad. We are interested in weak-NU operations compatible with $\mathbb{G}$. In this section we translate the question if $\mathbb{G}$ has a compatible r-ary weak-NU into a question whether there exists a weak-NU compatible with a certain family of digraphs on the set $\mathrm{Low} \cup \mathrm{Upp}$ which we denote by $\mathrm{Comp}(\mathbb{G})$. This construction will significantly simplify the proof of Theorem 4.4.

Definition 5.1. Let $\mathcal{I} \subseteq \mathrm{Paths}(\mathbb{G})$ and $\mathbb{S} \in \mathrm{Paths}(\mathbb{G})$. By a *compatible mapping* of $\mathcal{I}$ to $\mathbb{S}$ we mean a mapping $f : \Pi_{\mathbb{P}\in\mathcal{I}} V(\mathbb{P}) \to V(\mathbb{G})$ satisfying the following conditions:

(i) If $c_{\mathbb{P}}, d_{\mathbb{P}} \in V(\mathbb{P})$, $c_{\mathbb{P}} \xrightarrow{\mathbb{P}} d_{\mathbb{P}}$ for each $\mathbb{P} \in \mathcal{I}$ and all the vertices $c_{\mathbb{P}}$ have the same level, then $f(\langle c_{\mathbb{P}} : \mathbb{P} \in \mathcal{I}\rangle) \xrightarrow{\mathbb{G}} f(\langle d_{\mathbb{P}} : \mathbb{P} \in \mathcal{I}\rangle)$.

(ii) Let $a_{\mathbb{P}}$ and $b_{\mathbb{P}}$ be the initial and terminal vertices of $\mathbb{P} \in \mathcal{I}$, respectively. Then $f(\langle a_{\mathbb{P}} : \mathbb{P} \in \mathcal{I}\rangle)$ is the initial vertex of $\mathbb{S}$ and $f(\langle b_{\mathbb{P}} : \mathbb{P} \in \mathcal{I}\rangle)$ is its terminal vertex.

We say that $\mathcal{I}$ *maps compatibly* to $\mathbb{S}$ (via f) if there exist a compatible mapping f of $\mathcal{I}$ to $\mathbb{S}$.

Remark. Note that $\mathcal{I}$ maps compatibly to $\mathbb{S}$ if and only if there exists a homomorphism from the component of connectivity of the digraph $\Pi_{\mathbb{P}\in\mathcal{I}}\mathbb{P}$ containing the tuple of initial vertices $\langle a_{\mathbb{P}} : \mathbb{P} \in \mathcal{I}\rangle$ to $\mathbb{S}$. Indeed, if f is such a homomorphism, then we can extended it to $\Pi_{\mathbb{P}\in\mathcal{I}} V(\mathbb{P})$ by setting $f(c_1, \dots, c_{|\mathcal{I}|}) = c_1$ whenever $\langle c_1, \dots, c_{|\mathcal{I}|}\rangle$ is not in the above mentioned component. The property (i) is obvious and it is not hard to prove that (ii) holds as well (see Lemma 2.1 from [3]).

Definition 5.2. For each subset $\mathcal{I} \subseteq \mathrm{Paths}(\mathbb{G})$ we define a digraph $\mathbb{G}_{\mathcal{I}}$ as follows: Let $V(\mathbb{G}_{\mathcal{I}}) = \mathrm{Low} \cup \mathrm{Upp}$ and for $a, b \in V(\mathbb{G}_{\mathcal{I}})$, $a \xrightarrow{\mathbb{G}_{\mathcal{I}}} b$ iff $a \in \mathrm{Low}$, $b \in \mathrm{Upp}$, a and b are connected in $\mathbb{G}$ via $\mathbb{S} \in \mathrm{Paths}(\mathbb{G})$ and $\mathcal{I}$ maps compatibly to $\mathbb{S}$.

We define $\mathrm{Comp}(\mathbb{G}) = \{\mathbb{G}_{\mathcal{I}} : \mathcal{I} \subseteq \mathrm{Paths}(\mathbb{G})\}$ We say that an operation on the set $\mathrm{Low} \cup \mathrm{Upp}$ is *compatible with* $\mathrm{Comp}(\mathbb{G})$ (or that it is a *compatible operation of* $\mathrm{Comp}(\mathbb{G})$) if it is compatible with all $\mathbb{G}_{\mathcal{I}} \in \mathrm{Comp}(\mathbb{G})$.

In the next lemma we state several properties of $\mathrm{Comp}(\mathbb{G})$ which will be needed later:

Lemma 5.3.

(i) *Let f be an r-ary operation compatible with $\mathbb{G}$. Then for any $\mathcal{I} = \{\mathbb{S}_1, \mathbb{S}_2, \dots, \mathbb{S}_r\} \subseteq \mathrm{Paths}(\mathbb{G})$ we have $f(a_1, \dots, a_r) \xrightarrow{\mathbb{G}_{\mathcal{I}}} f(b_1, \dots, b_r)$, where a_i and b_i are the initial and terminal vertices of $\mathbb{S}_i$, respectively.*

(ii) *For any* $\mathcal{I} \subseteq \mathrm{Paths}(\mathbb{G})$ *and* $\mathbb{S} \in \mathcal{I}$ *we have* $a \overset{\mathbb{G}_{\mathcal{I}}}{\to} b$*, where* a *and* b *are the initial and terminal vertex of* $\mathbb{S}$*, respectively.*
(iii) *If* $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathrm{Paths}(\mathbb{G})$*, then* $\mathbb{G}_{\mathcal{I}}$ *is a subgraph of* $\mathbb{G}_{\mathcal{J}}$*.*

Proof. Statement (i) is an easy consequence of the compatibility of f with $\mathbb{G}$ and (ii) follows by an application of (i) to a projection operation, which is indeed compatible with $\mathbb{G}$.

To prove (iii), choose a, b with $a \overset{\mathbb{G}_{\mathcal{I}}}{\to} b$ and let $\mathbb{S}$ be the path connecting a to b in $\mathbb{G}$. Let f be a compatible mapping of $\mathcal{I}$ to $\mathbb{S}$. If we denote $\mathcal{I} = \{\mathbb{S}_1, \mathbb{S}_2, \ldots, \mathbb{S}_r\}$, then for each $i, 1 \leq i \leq r$ we have that $\mathbb{S}_i \in \mathcal{J}$, and thus by (ii) there exists a compatible mapping f_i of $\mathcal{J}$ to $\mathbb{S}_i$. Now the mapping $g(x_1, x_2, \ldots, x_r) = f(f_1(x_1), f_2(x_2), \ldots, f_r(x_r))$ is easily seen to be a compatible mapping of $\mathcal{J}$ to $\mathbb{G}$, proving that $a \overset{\mathbb{G}_{\mathcal{J}}}{\to} b$. □

We will prove that a special polyad $\mathbb{G}$ has an r-ary compatible weak-NU if and only if $\mathrm{Comp}(\mathbb{G})$ does. Let us first introduce some useful notation:

Definition 5.4. For any tuple $\bar{a} = \langle a_1, \ldots, a_r \rangle \in V(\mathbb{G})^r$ and $0 \leq l \leq \mathrm{hgt}(\mathbb{G})$ we define $\mathrm{Level}_l(\bar{a})$ to be the number

$$\mathrm{Level}_l(\bar{a}) = |\{i : a_i \text{ has level } l\}|.$$

Notice that $\mathrm{Level}_0(\bar{a}) = |\{i : a_i \in \mathrm{Low}\}|$ and $\mathrm{Level}_{\mathrm{hgt}(\mathbb{G})}(\bar{a}) = |\{i : a_i \in \mathrm{Upp}\}|$.

Theorem 5.5. *Let* $\mathbb{G}$ *be a special polyad.* $\mathbb{G}$ *has a compatible* r*-ary weak-NU iff there exists an* r*-ary weak-NU compatible with* $\mathrm{Comp}(\mathbb{G})$*.*

Proof. First, let ω be an r-ary weak-NU compatible with $\mathbb{G}$. We define an r-ary operation ω' on the set $\mathrm{Low} \cup \mathrm{Upp}$ as follows: Let $\bar{a} = \langle a_1, \ldots, a_r \rangle \in (\mathrm{Low} \cup \mathrm{Upp})^r$.

(1) If $\mathrm{Level}_l(\bar{a}) = r$ for some $l \in \{0, \mathrm{hgt}(\mathbb{G})\}$, we put $\omega'(\bar{a}) = \omega(\bar{a})$.
(2) If $\mathrm{Level}_l(\bar{a}) = r - 1$ for some $l \in \{0, \mathrm{hgt}(\mathbb{G})\}$, we put $\omega'(\bar{a}) = a_i$, where $a_i \in \mathrm{Upp}$ if $l = 0$ and $a_i \in \mathrm{Low}$ if $l = \mathrm{hgt}(\mathbb{G})$.
(3) In all other cases we put $\omega'(\bar{a}) = a_1$.

Claim. ω' is a weak-NU operation.

Let $a, b \in \mathrm{Low} \cup \mathrm{Upp}$ be arbitrary. We want to prove that $\omega'(a, \ldots, a, b) = \omega'(a, \ldots, a, b, a) = \cdots = \omega'(b, a, \ldots, a)$. Clearly, for all of these tuples the same case applies, one of the cases (1), (2). In case (1) it holds because ω is a weak-NU. In case (2) the definition is independent of the order of $a_1, \ldots, a_r$, so the desired property holds as well.

Claim. ω' is compatible with $\mathrm{Comp}(\mathbb{G})$.

Let $\mathcal{I} \subseteq \mathrm{Paths}(\mathbb{G})$ be arbitrary. Choose $\bar{a}, \bar{b} \in (\mathrm{Low} \cup \mathrm{Upp})^r$ such that $a_i \overset{\mathbb{G}_{\mathcal{I}}}{\to} b_i$ for each i. It follows that for both $\omega'(\bar{a})$ and $\omega'(\bar{b})$ case (1) of the definition applies. Thus we only need to establish $\omega(\bar{a}) \overset{\mathbb{G}_{\mathcal{I}}}{\to} \omega(\bar{b})$, which follows easily from Lemma 5.3 (i) and (iii).

To prove the other implication, assume that $\mathrm{Comp}(\mathbb{G})$ has a compatible r-ary weak-NU ω'. For each $\mathcal{I} \subseteq \mathrm{Paths}(\mathbb{G})$ and $\mathbb{S} \in \mathrm{Paths}(\mathbb{G})$, if $\mathcal{I}$ maps compatibly to $\mathbb{S}$, then we choose a fixed compatible mapping $f_{\mathcal{I},\mathbb{S}}$ of $\mathcal{I}$ to $\mathbb{S}$.

We will define an r-ary operation ω on $V(\mathbb{G})$. Choose arbitrary $\bar{a} \in V(\mathbb{G})^r$.

(1) If $\mathrm{Level}_l(\bar{a}) = r$ for some l, $0 \leq l \leq \mathrm{hgt}(\mathbb{G})$, then
 (1.1) if $l \in \{0, \mathrm{hgt}(\mathbb{G})\}$, we put $\omega(\bar{a}) = \omega'(\bar{a})$
 (1.2) else we define $\omega(\bar{a})$ below.

(2) If there exist $l \neq l'$ such that $\mathrm{Level}_l(\bar{a}) = 1$ and $\mathrm{Level}_{l'}(\bar{a}) = r - 1$, then we put $\omega(\bar{a}) = a_i$, where i is such that a_i has level l.

(3) In all other cases we put $\omega(\bar{a}) = a_1$.

In case (1.2) we define $\omega(\bar{a})$ as follows: For each $i = 1, \ldots, r$, let $\mathbb{S}_i \in \mathrm{Paths}(\mathbb{G})$ be such that $a_i \in V(\mathbb{S}_i)$. Denote by u_i and v_i the initial and terminal vertices of $\mathbb{S}_i$, respectively. Let a'_i be the vertex from $V(\mathbb{S}_i) \cap \{a_1, \ldots, a_r\}$ with minimal distance from u_i.

For $\mathcal{I} = \{\mathbb{S}_1, \ldots, \mathbb{S}_r\}$ we have $u_i \overset{\mathbb{G}_\mathcal{I}}{\to} v_i$ for all i; thus $\omega'(\bar{u}) \overset{\mathbb{G}_\mathcal{I}}{\to} \omega'(\bar{v})$. Denote by $\mathbb{S}$ the path connecting $\omega'(\bar{u})$ to $\omega'(\bar{v})$ in $\mathbb{G}$. Finally, we are ready to define

$$\omega(\bar{a}) = f_{\mathcal{I},\mathbb{S}}(\langle a'_i : \mathbb{S}_i \in \mathcal{I} \rangle).$$

and the definition of ω is completed.

Claim. ω is a weak-NU operation.

Choose arbitrary $a, b \in V(\mathbb{G})$. Again, the same case of the definition applies for $\omega(a, \ldots, a, b), \omega(a, \ldots, a, b, a), \ldots, \omega(b, a, \ldots, a)$, either (1) or (2). Similarly as before, in case (1.1) the weak-NU property holds because ω' is a weak-NU and in case (2) the definition is independent of the order of $a_1, \ldots, a_r$.

It remains to verify the weak-NU property for case (1.2). In this case, the definition of $\omega'(\bar{a})$ depends only on $\mathcal{I}$, $\mathbb{S}$ and $\langle a'_i : \mathbb{S}_i \in \mathcal{I} \rangle$. Clearly, if we switched a_i with a_j for some i, j, neither $\mathcal{I}$ nor $\langle a'_i : \mathbb{S}_i \in \mathcal{I} \rangle$ would change; thus they are independent of the order of $a_1, \ldots, a_r$. The path $\mathbb{S}$ depends only on $\omega'(\bar{u})$ and $\omega'(\bar{v})$. But since ω' is a weak-NU, $\bar{u}$ and $\bar{v}$ do not depend on which $\bar{a} \in \{\langle a, \ldots, a, b \rangle, \langle a, \ldots, a, b, a \rangle, \ldots, \langle b, a, \ldots, a \rangle\}$ we choose. Thus we have proved that ω is a weak-NU.

Claim. ω is compatible with $\mathbb{G}$.

Let $\bar{a}, \bar{b} \in V(\mathbb{G})^r$ be such that $a_i \overset{\mathbb{G}}{\to} b_i$ for $i = 1, \ldots, r$. It is easy to see that if $\omega(\bar{a})$ is defined by (2) or (3), then $\omega(\bar{b})$ is defined by the same case and $\omega(\bar{a}) \overset{\mathbb{G}}{\to} \omega(\bar{b})$. Let $\omega(\bar{a})$ be defined by (1.1), implying that $\omega(\bar{a}) = \omega'(\bar{a})$. Then $\omega(\bar{b})$ is defined by (1.2); and so $\omega(\bar{b}) = f_{\mathcal{I},\mathbb{S}}(\langle b_i : \mathbb{S}_i \in \mathcal{I} \rangle)$ for some $\mathcal{I}$, where $\mathbb{S}$ is a path with initial vertex $\omega(\bar{a})$. But since $\omega(\bar{b})$ lies on the path $\mathbb{S}$ and has level one, it follows that $\omega(\bar{a}) \overset{\mathbb{G}}{\to} \omega(\bar{b})$. If $\omega(\bar{a})$ is defined by (1.2) and $\omega(\bar{b})$ by (1.1), the proof is analogous.

It remains to investigate the case when both $\omega(\bar{a})$ and $\omega(\bar{b})$ are defined by (1.2). We have that $\omega(\bar{a}) = f_{\mathcal{I},\mathbb{S}}(\langle a'_i : \mathbb{S}_i \in \mathcal{I} \rangle)$ and $\omega(\bar{b}) = f_{\mathcal{I},\mathbb{S}}(\langle b'_i : \mathbb{S}_i \in \mathcal{I} \rangle)$, where $a'_i \overset{\mathbb{G}}{\to} b'_i$ for all i. It follows that $\omega(\bar{a}) \overset{\mathbb{G}}{\to} \omega(\bar{b})$, since $f_{\mathcal{I},\mathbb{S}}$ is a compatible mapping. □

In the proof of Theorem 4.4 we will need weak-NU operations compatible with $\mathrm{Comp}(\mathbb{G})$ having a certain property, which we call being *nice*. Luckily, as we prove in the next lemma, each weak-NU compatible with $\mathrm{Comp}(\mathbb{G})$ can be easily modified to have this property.

Definition 5.6. Let ω be an r-ary weak-NU compatible with $\mathrm{Comp}(\mathbb{G})$. We say that ω is *nice* if it satisfies the following: if $\{a_1, \ldots, a_r\} \subseteq \mathrm{Low} \setminus \{0\}$ and $\omega(\widehat{a}_1, \ldots, \widehat{a}_r) = \widehat{a}_i$ for some i, then $\omega(a_1, \ldots, a_r) = a_i$.

Lemma 5.7. *If* $\mathrm{Comp}(\mathbb{G})$ *has a compatible r-ary weak-NU, then it has a compatible r-ary nice weak-NU.*

Proof. Let ω be an r-ary weak-NU compatible with $\mathbb{G}$. We define ω' in the following way:

(1) If $\{a_1, \ldots, a_r\} \subseteq \mathrm{Low} \setminus \{0\}$ and $\omega(\widehat{a}_1, \ldots, \widehat{a}_r) = \widehat{a}_i$ for some i, then $\omega'(a_1, \ldots, a_r) = a_i$
(2) else we put $\omega'(\bar{a}) = \omega(\bar{a})$.

It is obvious that ω' is a nice weak-NU. Choose $\mathbb{G}_{\mathcal{I}} \in \mathrm{Comp}(\mathbb{G})$ and $a_i \in \mathrm{Low}$, $b_i \in \mathrm{Upp}$ such that $a_i \stackrel{\mathbb{G}_{\mathcal{I}}}{\to} b_i$, $i = 1, 2, \ldots r$. To verify that ω' is compatible with $\mathbb{G}_{\mathcal{I}}$ it suffices to consider the case when $\omega'(\bar{a})$ is defined by (1). But then for some i,

$$\omega'(a_1, \ldots, a_r) = a_i \stackrel{\mathbb{G}_{\mathcal{I}}}{\to} \widehat{a}_i = \omega(\widehat{a}_1, \ldots, \widehat{a}_r) = \omega'(\widehat{a}_1, \ldots, \widehat{a}_r),$$

which concludes the proof. □

Corollary 5.8. *Let $\mathbb{G}$ be a special polyad. $\mathbb{G}$ has a compatible r-ary weak-NU iff* $\mathrm{Comp}(\mathbb{G})$ *has a compatible r-ary nice weak-NU.*

Proof. Follows directly from Theorem 5.5 and Lemma 5.7. □

6. Proof of Theorem 4.4

Finally, we are ready to start proving the main result. Let $\mathbb{G}$ be a special polyad. According to Lemma 4.3, we may assume that $\mathbb{G}$ is a core. If $\mathbb{G}$ has no compatible weak-NU operation, then $\mathrm{CSP}(\mathbb{G})$ is NP-complete by Theorem 3.3. In this section we prove that if $\mathbb{G}$ has a compatible r-ary weak-NU, then it also has a compatible $(r+1)$-ary weak-NU.

We will use the translation of the problem presented in the previous section. Suppose that $\mathrm{Comp}(\mathbb{G})$ has a compatible r-ary nice weak-NU ω. Recall that by $\circ_\omega$ we mean the binary operation defined by

$$x \circ_\omega y = \omega(x, \ldots, x, y).$$

In the following two lemmata we present two different constructions of an $(r+1)$-ary compatible weak-NU from ω, imposing certain properties on $\circ_\omega$. Consequently, in the Proof of Theorem 4.4 we will quite easily show that each nice weak-NU compatible with $\mathrm{Comp}(\mathbb{G})$ has one of these properties.

Lemma 6.1. *If* $\mathrm{Comp}(\mathbb{G})$ *has a compatible r-ary nice weak-NU ω satisfying*

$$(\forall a \in \mathrm{Low})\ 0 \circ_\omega a = 0$$

then it has a compatible $(r+1)$-ary nice weak-NU.

Proof. We will prove that $\mathrm{Comp}(\mathbb{G})$ has a compatible $(r+1)$-ary weak-NU. Then by Lemma 5.7 it also has a nice one. We put

$$\mathrm{Maj} = \{a \in \mathrm{Low} : a \circ_\omega 0 = a\}.$$

First, we need to prove the following observation:

Claim. If $a \in \mathrm{Maj}$ and $b \in \mathrm{Low}$, then $a \circ_\omega b = a$.

The assumptions of this lemma state that the claim is true for $a = 0$. It remains to prove the claim for $a \neq 0$. For $b = 0$ the claim follows from the definition of Maj. Assume that $b \neq 0$. We will prove that $\widehat{a} \circ_\omega \widehat{b} = \widehat{a}$. Since ω is compatible with $\mathrm{Comp}(\mathbb{G})$, it is indeed compatible with the digraph $\mathbb{H} = \mathbb{G}_{\mathrm{Paths}(\mathbb{G})}$. In this digraph we have that $a \xrightarrow{\mathbb{H}} \widehat{a}$ and $0 \xrightarrow{\mathbb{H}} \widehat{b}$; and so $a = a \circ_\omega 0 \xrightarrow{\mathbb{H}} \widehat{a} \circ_\omega \widehat{b}$. We conclude that $\widehat{a} \circ_\omega \widehat{b} = \widehat{a}$. Since ω is nice, it follows that $a \circ_\omega b = a$ and the claim is proved.

In order to prove that $\mathrm{Comp}(\mathbb{G})$ has a compatible $(r+1)$-ary weak-NU, we will define an $(r+1)$-ary operation τ on the set $\mathrm{Low} \cup \mathrm{Upp}$. For $\bar{a} = \langle a_1, \dots, a_r, a_{r+1}\rangle \in (\mathrm{Low} \cup \mathrm{Upp})^{r+1}$ we define $\tau(\bar{a})$ as follows:

(1) If $\mathrm{Level}_l(\bar{a}) = r$ for some $l \in \{0, \mathrm{hgt}(\mathbb{G})\}$, then
   (1.1) if $\bar{a} = \langle a, \dots, a, b\rangle$ for some $a, b \in \mathrm{Low}$, $a \notin \mathrm{Maj}$, we put $\tau(\bar{a}) = a \circ_\omega b$ and if $\bar{a} = \langle \widehat{a}, \dots, \widehat{a}, \widehat{b}\rangle$ for some $\widehat{a}, \widehat{b} \in \mathrm{Upp}$, $a \notin \mathrm{Maj}$, we put $\tau(\bar{a}) = \widehat{a} \circ_\omega \widehat{b}$
   (1.2) else we define $\tau(\bar{a}) = \omega(a_1, \dots, a_r)$.
(2) If $\mathrm{Level}_l(\bar{a}) = r - 1$ for some $l \in \{0, \mathrm{hgt}(\mathbb{G})\}$, we put $\omega'(\bar{a}) = a_i$, where $a_i \in \mathrm{Upp}$ if $l = 0$ and $a_i \in \mathrm{Low}$ if $l = \mathrm{hgt}(\mathbb{G})$.
(3) In all other cases we put $\omega'(\bar{a}) = a_1$.

Claim. τ is a weak-NU.

It is easily seen that the weak-NU property holds in the cases (2) and (3) of the definition. As for the case (1), we will verify the property for $a, b \in \mathrm{Low}$. For $\widehat{a}, \widehat{b} \in \mathrm{Upp}$ we can proceed analogously.

If $a \in \mathrm{Maj}$, then case (1.2) applies. We get $\tau(a, \dots, a, b) = \omega(a, \dots, a) = a$, while $\tau(a, \dots, a, b, a) = \dots = \tau(b, a, \dots, a) = a \circ_\omega b = a$; and so the weak-NU property holds.

Now, suppose that $a \notin \mathrm{Maj}$. Then $\tau(a, \dots, a, b) = a \circ_\omega b$ by (1.1) and $\tau(a, \dots, a, b, a) = \dots = \tau(b, a, \dots, a) = a \circ_\omega b$ by (1.2). We conclude that τ is indeed a weak-NU operation.

Claim. τ is compatible with $\mathrm{Comp}(\mathbb{G})$.

Choose arbitrary $\mathbb{G}_\mathcal{I} \in \mathrm{Comp}(\mathbb{G})$ and $\bar{a}, \bar{b} \in (\mathrm{Low} \cup \mathrm{Upp})^{r+1}$ such that $a_i \xrightarrow{\mathbb{G}_\mathcal{I}} b_i$, $i = 1, 2, \dots r$. If the same cases of the definition apply for $\tau(\bar{a})$ and $\tau(\bar{b})$, then the compatibility condition follows from compatibility of the operations $\circ_\omega$ in case (1.1), ω in (1.2) and the projection operations in cases (2) and (3).

It can be easily seen that the only case when $\tau(\bar{a})$ and $\tau(\bar{b})$ are defined by different cases of the definition is when $\tau(\bar{a})$ is defined by (1.2) and $\tau(\bar{b})$ is defined by (1.1). In this situation we have that $\bar{b} = \langle \widehat{c}, \dots, \widehat{c}, \widehat{d}\rangle$ for some $\widehat{c}, \widehat{d} \in \mathrm{Upp}$, $c \notin \mathrm{Maj}$ and $\tau(\bar{b}) = \widehat{c} \circ_\omega \widehat{d}$. Since $a_i \xrightarrow{\mathbb{G}_\mathcal{I}} \widehat{c}$ for $i = 1, \dots, r$, we get $\tau(\bar{a}) = \omega(a_1, \ldots, a_r) \stackrel{\mathbb{G}_{\mathcal{I}}}{\to} \omega(\widehat{c}, \ldots, \widehat{c}) = \widehat{c}$; and so $\omega(a_1, \ldots, a_r) \in \{0, c\}$. We also know that $0 \in \{a_1, \ldots, a_r\}$, as otherwise case (1.1) would apply for $\tau(\bar{a})$.

First, let $\omega(a_1, \ldots, a_r) = 0$. Since $0 \stackrel{\mathbb{G}_{\mathcal{I}}}{\to} \widehat{c}$ and $a_{r+1} \stackrel{\mathbb{G}_{\mathcal{I}}}{\to} \widehat{d}$, from the compatibility of $\circ_\omega$ we obtain

$$\tau(\bar{a}) = \omega(a_1, \ldots, a_r) = 0 = 0 \circ_\omega a_{r+1} \stackrel{\mathbb{G}_{\mathcal{I}}}{\to} \widehat{c} \circ_\omega \widehat{d} = \tau(\bar{b}),$$

proving the compatibility condition for τ in this case.

Second, assume that $\omega(a_1, \ldots, a_r) = c$. Notice that $c \in \{a_1, \ldots, a_r\}$ (as $\omega(0, \ldots, 0) = 0$), implying that $c \stackrel{\mathbb{G}_{\mathcal{I}}}{\to} \widehat{c}$. We will prove that $\widehat{c} \circ_\omega \widehat{d} = \widehat{c}$. Then it will follow that

$$\tau(\bar{a}) = \omega(a_1, \ldots, a_r) = c \stackrel{\mathbb{G}_{\mathcal{I}}}{\to} \widehat{c} = \widehat{c} \circ_\omega \widehat{d} = \tau(\bar{b}),$$

which will conclude the proof.

In order to prove that $\widehat{c} \circ_\omega \widehat{d} = \widehat{c}$, consider again the digraph $\mathbb{H} = \mathbb{G}_{\mathrm{Paths}(\mathbb{G})}$. Let $j \in \{1, \ldots, r\}$ be such that $a_j = 0$. We have that $a_j \stackrel{\mathbb{H}}{\to} \widehat{d}$ and $a_i \stackrel{\mathbb{H}}{\to} \widehat{c}$ for all $i \in \{1, \ldots, r\}$. From the compatibility of ω with $\mathbb{H}$ it follows that

$$c = \omega(a_1, \ldots, a_r) \stackrel{\mathbb{H}}{\to} \omega(\widehat{c}, \ldots, \widehat{c}, \widehat{d}, \widehat{c}, \ldots, \widehat{c}) = \widehat{c} \circ_\omega \widehat{d}$$

implying that $\widehat{c} \circ_\omega \widehat{d} = \widehat{c}$; and the proof is finished. $\square$

In the next construction we define a partial order on the set Low$\cup$Upp and then use the binary operation $\circ_\omega$ to "compare the incomparable" elemets (see the remark below).

Lemma 6.2. *If* Comp$(\mathbb{G})$ *has a compatible* r*-ary nice weak-NU* ω *satisfying*

$$(\exists z \in \mathrm{Low})(\forall a \in \mathrm{Low}, a \neq z)\ a \circ_\omega 0 = 0 \circ_\omega a = 0$$

then it has a compatible $(r+1)$*-ary nice weak-NU.*

Proof. Again, it suffices to prove that Comp$(\mathbb{G})$ has a compatible $(r+1)$-ary weak-NU and then apply Lemma 5.7. We can assume that $z \neq 0$, since for $z = 0$ we even have that $a \circ_\omega 0 = 0 \circ_\omega a = 0$ for all $a \in \mathrm{Low}$, which allows us to take any $z' \in \mathrm{Low}$ instead of z.

We will define a partial order $\preceq$ on the set $\mathrm{Low} \cup \mathrm{Upp}$. For all $\widehat{c} \in \mathrm{Upp}$, $\widehat{c} \neq \widehat{z}$ we put $z \prec \widehat{z} \prec 0 \prec \widehat{c}$ if $\widehat{c} \in \mathrm{Half}$ and $z \prec \widehat{z} \prec 0 \prec \widehat{c} \prec c$ else. We define $\preceq$ to be the partial order generated by these relations.

Let us fix an arbitrary linear order $<$ of the set $\mathrm{Upp} \setminus \{\widehat{z}\}$. (We can assume without loss of generality that $z = 1$ and $\mathrm{Upp} \setminus \{\widehat{z}\} = \{\widehat{2} < \widehat{3} < \cdots < \widehat{n}\}$.)

For each $i = 1, 2, \ldots, n-1$ we denote by t_i the i-ary operation defined in the following way:

$$\begin{aligned}
&t_1(x) = x \\
&t_2(x_1, x_2) = x_1 \circ_\omega x_2 \\
&\vdots \\
&t_i(x_1, \ldots, x_i) = t_{i-1}(x_1, \ldots, x_{i-1}) \circ_\omega x_i
\end{aligned}$$

Note that all these operations are compatible with Comp$(\mathbb{G})$.

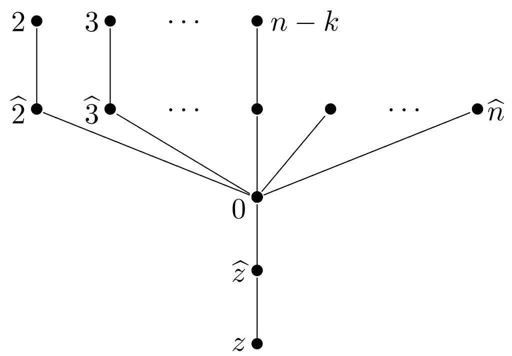

FIGURE 4. The partial order $\preceq$

For any $\bar{a} \in (\mathrm{Low} \cup \mathrm{Upp})^{r+1}$ we define the set $S(\bar{a})$ to be the smallest subset of $\mathrm{Low} \cup \mathrm{Upp}$ containing $\{a_1, \ldots, a_{r+1}\}$ and closed under all the operations t_i (i.e., $t_i(c_1, \ldots, c_i) \in S(\bar{a})$ for all $i = 1, \ldots, n-1$ and $c_1, \ldots, c_i \in S(\bar{a})$).

For each $\widehat{c} \in \mathrm{Upp}$ we define the set $R(\widehat{c})$ in the following way: we put $R(\widehat{c}) = \{\widehat{c}\}$ if $\widehat{c} \in \mathrm{Half}$ and $R(\widehat{c}) = \{\widehat{c}, c\}$ else. Now, $\tau(\bar{a})$ is defined as follows:

(1) If $S(\bar{a})$ has the least element with respect to $\preceq$, we define $\tau(\bar{a})$ to be that element
(2) else let $\{\widehat{c_1} < \widehat{c_2} < \cdots < \widehat{c_m}\}$ be the set of all $\widehat{c} \in \mathrm{Upp} \setminus \{\widehat{z}\}$ such that $S(\bar{a}) \cap R(\widehat{c}) \neq \emptyset$. Note that $m \geq 2$. For $i = 1, \ldots, m$ we denote by a'_i the $\preceq$-least element of $S(\bar{a}) \cap R(\widehat{c_i})$. Finally, we put $\tau(\bar{a}) = t_m(a'_1, a'_2, \ldots, a'_m)$.

It is easy to check the weak-NU property of τ, since $\tau(\bar{a})$ is independent of the order of $a_1, \ldots, a_{r+1}$ (in both cases of the definition). It remains to prove that τ is compatible with $\mathrm{Comp}(\mathbb{G})$.

Choose $\mathbb{G}_{\mathcal{I}} \in \mathrm{Comp}(\mathbb{G})$ and $\bar{a}, \bar{b} \in (\mathrm{Low} \cup \mathrm{Upp})^{r+1}$ such that $a_i \overset{\mathbb{G}_{\mathcal{I}}}{\to} b_i$, $i = 1, 2, \ldots r$. First, let both $\tau(\bar{a})$ and $\tau(\bar{b})$ be defined by case (1). Observe that for each $c \in S(\bar{a})$ there exists $d \in S(\bar{b})$ such that $c \overset{\mathbb{G}_{\mathcal{I}}}{\to} d$ and conversely, for each $d' \in S(\bar{b})$ there exists $c' \in S(\bar{a})$ with $c' \overset{\mathbb{G}_{\mathcal{I}}}{\to} d'$. From this fact it follows easily that if c and d are the $\preceq$-least elements of $S(\bar{a})$ and $S(\bar{b})$, respectively, then $c \overset{\mathbb{G}_{\mathcal{I}}}{\to} d$.

If $\tau(\bar{a})$ and $\tau(\bar{b})$ are both defined by case (2), the compatibility can be verified similarly: since $0 \notin S(\bar{a})$ (otherwise $S(\bar{a})$ would have a $\preceq$-least element), we get that for each $c \in \mathrm{Low} \setminus \{0, z\}$, $c \in S(\bar{a})$ iff $\widehat{c} \in S(\bar{b})$, and if it is the case, then $c \overset{\mathbb{G}_{\mathcal{I}}}{\to} \widehat{c}$. From this fact and the compatibility of the operations t_i, it follows directly that $\tau(\bar{a}) \overset{\mathbb{G}_{\mathcal{I}}}{\to} \tau(\bar{b})$.

It is easily seen that if $\tau(\bar{a})$ is defined by case (2), then so is $\tau(\bar{b})$. Thus it only remains to investigate the case when $\tau(\bar{a})$ is defined by (1) and $\tau(\bar{b})$ by (2). In this case, we have that $\tau(\bar{a}) = 0$ and $\tau(\bar{b}) = t_m(\widehat{c_1}, \ldots, \widehat{c_m})$ for some $2 \leq m \leq n-1$ and $\widehat{c_i} \in \mathrm{Upp} \setminus \{\widehat{z}\}$.

For each i, let $c_i' \in S(\bar{a})$ be $\preceq$-minimal such that $c_i' \overset{\mathbb{G}_{\mathcal{I}}}{\to} \widehat{c_i}$. (Note that $c_i' = 0$ if $\widehat{c_i} \in \mathrm{Half}$ and $c_i' \in \{0, c_i\}$ else.) Since $0 \in S(\bar{a})$, there exists j such that $c_j' = 0$. We will prove that $t_m(c_1', \ldots, c_m') = 0$. Then the proof will conclude, as we will have that

$$\tau(\bar{a}) = 0 = t_m(c_1', \ldots, c_m') \overset{\mathbb{G}_{\mathcal{I}}}{\to} t_m(\widehat{c}_1, \ldots, \widehat{c}_m) = \tau(\bar{b}).$$

In order to establish $t_m(c_1', \ldots, c_m') = 0$, we first need to prove that $t_j(c_1', \ldots, c_{j-1}', c_j') = t_{j-1}(c_1', \ldots, c_{j-1}') \circ_\omega 0 = 0$. Since the $\preceq$-least element of $S(\bar{a})$ is 0, it follows that $t_{j-1}(c_1', \ldots, c_{j-1}') \neq z$. But then, by the assumptions of the lemma, $t_{j-1}(c_1', \ldots, c_{j-1}') \circ_\omega 0 = 0$. The rest is easy. We have that

$$t_{j+1}(c_1', \ldots, c_{j+1}') = t_j(c_1', \ldots, c_j') \circ_\omega c_{j+1}' = 0 \circ_\omega c_{j+1}'$$

and since $c_{j+1}' \neq z$, it follows that $t_{j+1}(c_1', \ldots, c_{j+1}') = 0$. We can proceed by induction, proving that $t_m(c_1', \ldots, c_m') = 0$. $\square$

Remark. A totally symmetric operation is an operation f satisfying $f(\bar{a}) = f(\bar{a'})$ whenever $\{a_1, \ldots, a_r\} = \{a_1', \ldots, a_r'\}$. In fact, in the proof of Lemma 6.2 we constructed a totally symmetric idempotent operation. Moreover, we didn't need its arity to be $r+1$. Thus we can use the same construction to prove that if a special polyad has a compatible weak-NU ω satisfying

$$(\exists z \in \mathrm{Low})(\forall a \in \mathrm{Low}, a \neq z)\ a \circ_\omega 0 = 0 \circ_\omega a = 0,$$

then it has compatible totally symmetric idempotent operations of all arities. This property is also known to imply bounded width (see [5]). However, it is not the case with the construction in Lemma 6.1. Therefore we need Theorem 3.4 in its full strength.

Finally, we are ready to prove the main result.

Proof of Theorem 4.4.

Proof. Choose an arbitrary special polyad and let $\mathbb{G}$ be its core. By Lemma 4.3, $\mathbb{G}$ is also a special polyad. According to Theorem 3.3, if $\mathbb{G}$ has no compatible weak-NU operation, then CSP($\mathbb{G}$) is NP-complete. Assume that $\mathbb{G}$ has a compatible r_0-ary weak-NU operation. We will prove that then $\mathbb{G}$ has compatible weak-NUs of all arities $r \geq r_0$. It will then follow by Theorem 3.4 that $\mathbb{G}$ has bounded width. We will proceed by induction on r.

According to Corollary 5.8, it suffices to prove that if Comp($\mathbb{G}$) has a compatible r-ary nice weak-NU, then it has a compatible $(r+1)$-ary nice weak-NU. Let ω be an r-ary nice weak-NU compatible with Comp($\mathbb{G}$). If ω satisfies

$$(\forall a \in \mathrm{Low})\ 0 \circ_\omega a = 0,$$

then by Lemma 6.1, Comp($\mathbb{G}$) has a compatible $(r+1)$-ary nice weak-NU. Suppose that there exists $z \in \mathrm{Low}$ such that $0 \circ_\omega z \neq 0$. Consider the digraph $\mathbb{H} = \mathbb{G}_{\mathrm{Paths}(\mathbb{G})} \in \mathrm{Comp}(\mathbb{G})$. We have $0 \overset{\mathbb{H}}{\to} \widehat{z}$ and $z \overset{\mathbb{H}}{\to} \widehat{z}$. Since ω is compatible with $\mathbb{H}$, we obtain $0 \circ_\omega z \overset{\mathbb{H}}{\to} \widehat{z} \circ_\omega \widehat{z} = \widehat{z}$ implying that $0 \circ_\omega z = z$.

We will prove that $a \circ_\omega 0 = 0 \circ_\omega a = 0$ for all $a \in \mathrm{Low}, a \neq z$. This will finish the proof, since then according to Lemma 6.2, Comp($\mathbb{G}$) has a compatible $(r+1)$-ary nice weak-NU.

The statement holds for $a = 0$. Striving for a contradiction, suppose that there exists $a \in \mathrm{Low} \setminus \{0, z\}$ such that $0 \circ_\omega a = a$ or $a \circ_\omega 0 = a$. As $a \stackrel{\mathbb{H}}{\to} \widehat{a}$, $0 \stackrel{\mathbb{H}}{\to} \widehat{a}$ and $0 \stackrel{\mathbb{H}}{\to} \widehat{z}$, we obtain in both cases that $a \stackrel{\mathbb{H}}{\to} \widehat{a} \circ_\omega \widehat{z}$ (since $0 \circ_\omega a = \omega(0, 0, \ldots, 0, a) \stackrel{\mathbb{H}}{\to} \omega(\widehat{z}, \widehat{a}, \widehat{a} \ldots, \widehat{a}) = \widehat{a} \circ_\omega \widehat{z}$). It follows that $\widehat{a} \circ_\omega \widehat{z} = \widehat{a}$. On the other hand, from $0 \circ_\omega z = z$ we can obtain in a similar way that $\widehat{a} \circ_\omega \widehat{z} = \widehat{z}$. It is a contradiction, since we assumed that $a \neq z$. $\square$

Acknowledgements. The author wishes to thank his supervisor Libor Barto for valuable comments, support and encouragement.

References

[1] L. Barto, M. Kozik. *Constraint Satisfaction Problems of Bounded Width.* Manuscript.
[2] L. Barto, M. Kozik and T. Niven. *The CSP dichotomy holds for digraphs with no sources and no sinks (a positive answer to a conjecture of Bang-Jensen and Hell).* SIAM Journal on Computing 38/5 (2009), 1782-1802.
[3] L. Barto, M. Kozik, M. Maróti and T. Niven. *CSP dichotomy for special triads.* To appear in Proceedings of the AMS.
[4] A. Bulatov, P. Jeavons and A. Krokhin. *Classifying the complexity of constraints using finite algebras.* SIAM J. Comput., 34(3):720-742 (electronic), 2005.
[5] V. Dalmau and J. Pearson. *Set functions and width 1 problems.* Proceedings of the 5th International Conference on Constraint Programming (CP99), Lecture Notes in Comput. Sci. 1713, Springer-Verlag, Berlin, 1999, pp. 159-173.
[6] T. Feder. *Classification of homomorphisms to oriented cycles and of k-partite satisfiability.* SIAM J. Discrete Math., 14(4):471-480 (electronic), 2001.
[7] T. Feder and M. Vardi. *The computational structure of monotone monadic SNP and constraint satisfaction: a study through Datalog and group theory.* SIAM J. Comput., 28(1):57-104 (electronic), 1999.
[8] W. Gutjahr, E. Welzl and G. Woeginger. *Polynomial graph-colorings.* Discrete Appl. Math., 35(1):29-45, 1992.
[9] P. Hell and J. Nešetřil. *On the complexity of H-coloring.* J. Combin. Theory Ser. B, 48(1):92-110, 1990.
[10] P. Jeavons, D. Cohen and M. Gyssens. *Closure properties of constraints.* J. ACM, 44(4):527-548, 1997.
[11] A. Krokhin, A. Bulatov and P. Jeavons. *The complexity of constraint satisfaction: An algebraic approach.* Structural Theory of Automata, Semigroups and Universal Algebra (Montreal, 2003), NATO Sci. Ser. II: Math., Phys., Chem., volume 207, 181-213, 2005.
[12] B. Larose and L. Zádori. *Bounded width problems and algebras.* Algebra Universalis, 56 no. 3-4, 439-466, 2007.
[13] M. Maróti and R. McKenzie. *Existence theorems for weakly symmetric operations.* Algebra Universalis 59 (2008), no. 34, 463-489.

Department of Algebra, Charles University, Prague, Czech Republic
E-mail address: jakub.bulin@gmail.com