LELEK'S CONJECTURE

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Abstract

Lelek's conjecture which states that metric continua with span zero are chainable has been one of the most widely investigated problems in continuum theory over the past 40 years. We broaden our field of interest to non-metric continua and prove that if there is a non-metric counterexample to Lelek's conjecture we can convert it to a metric one. For a continuum X we take the lattice of all of its closed subsets 2^X and consider a countable elementary sublattice L of 2^X that we represent by a metric continuum wL via the Wallman representation for distributive lattices. By means of set theory, we obtain an L such that X is not chainable if and only if wL is not chainable and X has span zero if and only if wL has span zero. In the proof of the latter we use Shelah's theorem stating that every two elementarily equivalent models have isomorphic ultrapowers.

Keywords: Chainability, span, elementarity

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1 Introduction

In the past 40 years, Lelek's conjecture which states that metric continua with span zero are chainable has been one of the most widely investigated problems in continuum theory.

In his article [21], A. Lelek describes the span for metric spaces as follows:

> "The span of a metric space is the least upper bound of numbers α such that, roughly speaking, two points can move over the same portion of the space keeping a distance at least α from each other. The surjective span is obtained if it is required that, in addition, the whole space be covered by each of the moving points. These geometric ideas turn out to be important in continua theory."

More formally, the span of a compact metric space (X, d) is the supremum of all numbers α such that there exists a subcontinuum Z of the square $X \times X$ projecting on both axes on the same set with $d(x, y) \geq \alpha$ for all $(x, y) \in Z$. Span was first defined by Lelek in [19] in a more general setting of mappings. In the same article, he proved that chainable implies span zero and the span zero implies atriodic and dimension 1. Later ([20]), he raised the following question

Question 1 (Lelek)**.** *Are metric continua having span zero chainable?*

A positive answer to this question would solve the classification problem of homogeneous plane continua ([27]).

Hence, much effort has been exerted to examine the span of metric continua. None of the properties observed so far is known to provide the answer to Lelek's question, although several weaker results have been obtained, many of them using weaker variants of span (the first three are due to Lelek, the last one is due to Davis)

- *semispan* - the condition on the subcontinuum Z in the definition of span is relaxed to one projection being a subset of the other one,

- *surjective span* - Z projects onto X in both coordinates
- *surjective semispan* - Z projects onto X in at least one coordinate
- *symmetric span* - Z is symmetric, which means that with every $(x, y) \in Z$ also $(y, x) \in Z$.

Using the null-homotopy of mappings from spaces with span zero to one-dimensional polyhedra, Lelek deduced that the continua of span zero are tree-like (an inverse limit of connected graphs; see [23]). However, not every atriodic tree-like continuum has span zero (see [16]).

Further, span zero is preserved by inverse limits, and continua with span zero are weakly chainable (a continuous image of the pseudoarc; see [28]).

(Surjective) span and (surjective) semispan are not equal in general ([22]). But (surjective) span zero and (surjective) semispan zero are equivalent ([8]), which is essential in many results on span (for instance [7, 9, 11]).

It is an open problem whether chainabiliy or span zero are preserved by confluent mappings. A mapping $f : X \to Y$ is confluent if every component of a preimage of a subcontinuum C of Y maps onto C. Oversteegen gave a partial answer to the second part of this problem in [26] by proving that if a hereditarily indecomposable continuum X has span zero and $f : X \to Y$ is a continuous surjective mapping whose square $f \times f$ is confluent, then Y has span zero as well. Duda ([10]) generalized the result to all continua under the condition that both f and its square $f \times f$ are confluent.

Striving for a counterexample to Question 1, we broaden our field of search to non-metric continua. In chapter 3, we show that if there is a non-metric counterexample, we can convert it to a metric one using a combination of topology, lattice theory, model theory and set theory. Having a continuum X, we consider a countable elementary sublattice L of the lattice 2^X of all closed subsets of X and assign to L a metric continuum wL that has a base for closed subsets isomorphic to L. The points of the space wL are ultrafilters on L and the closed sets for the topology are generated by the sets of the form $U(a)$ for $a \in L$, where $U(a)$ consists of the ultrafilters to which a belongs. This construction is due to Wallman ([31]). It is not surprising that the space wL shares many properties with X. However, we were not able to show that whenever X is non-chainable and of span zero, then so is wL. Luckily, restricting ourselves to "special" elementary sublattices, we achieve the desired result. To obtain the "special" elementary sublattices, we consider the sets $H(\theta)$ (see [18]) that model ZFC-P (Zermelo-Frankel set theory with the Axiom of Choice and without the Power Set Axiom) and take a countable elementary submodel $\mathcal{M}$ of $H(\theta)$ containing 2^X as its element. Then $L = \mathcal{M} \cap 2^X$ is a countable elementary sublattice of 2^X of which we know more than a general elementary sublattice. If $K = \mathcal{M} \cap 2^{X\times X}$, then $wL \times wL$ and wK are homeomorphic. Knowing that, we are ready to prove that if X is not chainable, neither is wL (Theorem 3.3.4, originally from [30]) and if X has span zero then so does wL (Theorem 3.3.6).

In the proof of Theorem 3.3.6 on reflection of span zero, we use the statement that every two elementarily equivalent models have isomorphic ultrapowers, which is a generalization of Keisler's theorem ([17]) made by Shelah ([29]). For the purpose of the continuum theory it was first used by Gurevič([13]). We apply it to an elementary embedding $f : K \to 2^{X\times X}$ for a continuum X, where $K = \mathcal{M} \cap 2^{X\times X}$ for an elementary submodel $\mathcal{M}$ of $H(\theta)$. Shelah's theorem provides an ultrafilter $\mathcal{U}$ and an isomorphism $h : \prod_{\mathcal{U}} K \to \prod_{\mathcal{U}} 2^{X\times X}$ such that $h \circ \Delta = \Delta \circ f$, where Δ denotes the elementary embedding of a structure into its ultrapower. Applying the Wallman's representation for lattices, which can be extended to lattice homomorphisms, we get an onto mapping $w(f) : X \times X \to wK$ and a homeomorphism $w(h) : w(\prod_{\mathcal{U}} 2^{X\times X}) \to w(\prod_{\mathcal{U}} K)$ such that $w(h) \circ w(\Delta) = w(\Delta) \circ w(f)$. The mapping $w(\Delta)$ is called a codiagonal map and denoted by ∇, and $w(\prod_{\mathcal{U}} 2^{X\times X})$ is called the ultracopower of $X \times X$ and denoted by $\sum_{\mathcal{U}} X \times X$ (widely studied by Bankston; [2, 3, 4]). We show that if $Z \subset wL \times wL = wK$ witnesses some kind of span non-zero in wL ($L = \mathcal{M} \cap 2^X$), then $\nabla \circ w(h)^{-1}[\sum_{\mathcal{U}} Z]$ witnesses the same kind of span non-zero in X.

2 Preliminaries

Lattice theory combined with model theory is recognized to be a powerful tool in the study of compact (Hausdorff) spaces. We are particularly interested in continua, by which we mean connected compact Hausdorff spaces. A space is connected if it cannot be written as a disjoint union of two non-empty open sets.

2.1 Lattices

Definition 2.1.1 (Lattice)**.** A partially ordered set $(L, \leq)$ is called a *lattice* if every two elements $a, b \in L$ have a supremum (also called join and denoted by $a \sqcup b$) and infimum (called meet and denoted by $a \sqcap b$) in L. A lattice is called bounded if it has a smallest and a largest element, usually denoted by $\mathbf{0}$ and $\mathbf{1}$ respectively. Every lattice can be made into a bounded lattice by adding the greatest and the least element to the lattice.

Definition 2.1.2 (Lattice algebra)**.** An algebra $(L, \sqcup, \sqcap)$, where $\sqcup$ and $\sqcap$ are binary operations, is called a *lattice algebra* if every three elements $a, b, c \in L$ satisfy the following equalities:

$$\begin{aligned} a \sqcup a &= a \\ a \sqcap a &= a \end{aligned} \tag{2.1}$$

$$\begin{aligned} a \sqcup b &= b \sqcup a \\ a \sqcap b &= b \sqcap a \end{aligned} \tag{2.2}$$

$$\begin{aligned} a \sqcup (b \sqcup c) &= (a \sqcup b) \sqcup c \\ a \sqcap (b \sqcap c) &= (a \sqcap b) \sqcap c \end{aligned} \tag{2.3}$$

$$a \sqcup (a \sqcap b) = a \sqcap (a \sqcup b) = a \tag{2.4}$$

A lattice algebra with constants $\mathbf{0}, \mathbf{1}$ satisfying

$$a \sqcup \mathbf{0} = a \text{ and } a \sqcap \mathbf{0} = \mathbf{0} \tag{2.5}$$

$$a \sqcup \mathbf{1} = \mathbf{1} \text{ and } a \sqcap \mathbf{1} = a \tag{2.6}$$

is called a bounded lattice algebra.

In other words, a (bounded) lattice algebra is a structure for the language $\{\sqcup, \sqcap\}$ ($\{\sqcup, \sqcap, \mathbf{0}, \mathbf{1}\}$) that models the universal closure of the formulae above.

Lemma 2.1.3. *Every (bounded) lattice satisfies the equalities in the definition of (bounded) lattice algebra. If L is a lattice algebra, then putting $a \leq b$ if and only if $a \sqcup b = b$ (or equivalently $a \sqcap b = a$), we obtain a lattice order '$\leq$' on L. If L is bounded, then $\mathbf{0}$ and $\mathbf{1}$ are the smallest and the largest elements of* $(L, \leq)$

In what follows, by lattice L we mean a bounded lattice $(L, \leq)$ (or equivalently a bounded lattice algebra $(L, \sqcup, \sqcap, \mathbf{0}, \mathbf{1})$).

Example 2.1.4. Let X be a topological space and let 2^X denote the set of all its closed subsets. Then 2^X with the operations union and intersection, and constants empty set and X forms a lattice. The order in this lattice is given by set inclusion. Every base for the closed sets in X closed under finite unions and finite intersections is an example of a sublattice of 2^X. In what follows we call such a base a *lattice base.*

If a lattice L has the following three properties, then we can find a compact Hausdorff space that has a lattice base isomorphic to L.

Definition 2.1.5 (Distributive lattice)**.** A lattice L is called *distributive* if for every triple $a, b, c \in L$ the following equation holds:

$$a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c), \tag{2.7}$$

or equivalently

$$a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c) \tag{2.8}$$

Definition 2.1.6 (Disjunctive lattice)**.** A lattice L is called *disjunctive* if it models the sentence

$$\forall ab\ \exists c\ (a \nleq b \to c \neq \mathbf{0} \wedge c \leq a \wedge b \sqcap c = \mathbf{0}).$$

Definition 2.1.7 (Normal lattice)**.** A lattice L is called *normal* if it models the sentence

$$\forall ab\ \exists cd\ (a \sqcap b = \mathbf{0} \to a \sqcap d = \mathbf{0} \wedge b \sqcap c = \mathbf{0} \wedge c \sqcup d = \mathbf{1}).$$

The underlying set of the space assigned to L will be the set of all ultrafilters on the lattice L.

Definition 2.1.8 ((Ultra)filter)**.**

- We call a non-empty subset $\mathcal{F}$ of a lattice L a *filter* on L if it satisfies the following conditions:

  (i) if $a \in \mathcal{F}$, $b \in L$ and $a \leq b$ then $b \in \mathcal{F}$,

  (ii) if $a, b \in \mathcal{F}$ then $a \sqcap b \in \mathcal{F}$.

- A filter on L is *proper* if it is not equal to L.

- A filter $\mathcal{F}$ is called *prime* if $a \sqcup b \in \mathcal{F}$ implies $a \in \mathcal{F}$ or $b \in \mathcal{F}$.

- A proper filter is called *maximal* (or an *ultrafilter*) when L contains no larger proper filters.

- For an element $a \in L$, the filter consisting of all elements of L that are larger or equal to a is called *principal*.

It is easy to see that every subset of a lattice closed under finite meets can be extended to an ultrafilter.

2.2 Wallman's representation theorem

In [31], H. Wallman extended the Stone duality between Boolean algebras and totally disconnected compact Hausdorff spaces, to distributive lattice and compact T_1-spaces. However, what he obtained is not a full duality, as it assigns to numerous lattices the same space.

Theorem 2.2.1. *Let L be a distributive lattice. Then there is a compact T_1-space X with a base for closed sets being a homomorphic image of L. If L is also disjunctive, then X has a base for closed sets isomorphic to L. X is Hausdorff if and only if L is normal.*

Proof. We take as the points of X the set of all ultrafilters on L and the sets $U(a) = \{x \in X : a \in x\}$ as a base B for the closed sets for the topology on X.

It is easy to see that for every $a, b \in L$ we have $U(a \sqcap b) = U(a) \cap U(b)$ and $U(a \sqcup b) = U(a) \cup U(b)$. Thus the map $U : L \to 2^X$ from L to the lattice of all closed subsets of X given by $a \mapsto U(a)$ is a homomorphism onto B. Let L be also disjunctive. For every $a \neq b$ either $a \nleq b$ or $b \nleq a$. Without loss of generality assume that $a \not\leq b$. From disjunctivity, there is $c \neq \mathbf{0}$ such that $c \leq a$ and $c \sqcap b = \mathbf{0}$. Any ultrafilter that contains c contains also a but cannot contain b, hence $U(a) \neq U(b)$. So U is one-to-one onto its image, which means that L and B are isomorphic.

We show that X is compact. Let B' be a subset of B having the finite intersection property. We have that $B' = \{U(a) \mid a \in L'\}$, for some $L' \subset L$. As $U(a \sqcap b) = U(a) \cap U(b)$, all finite meets of elements from L' are non-zero, hence there is an ultrafilter x extending L'. Then $x \in U(a)$ for every $a \in L'$, so $x \in \bigcap B' \neq \emptyset$.

Since $\{U(a) \mid a \in L\}$ is a base for closed sets, $\{x\} = \bigcap\{U(a) \mid a \in x\}$. For any two distinct points x and y in X, there are $a \in x \setminus y$ and $b \in y \setminus x$. It gives that $y \in X \setminus U(a)$ whilst $x \in U(a)$ and $x \in X \setminus U(b)$ whilst $y \in U(b)$. Thus X is T_1.

Let L be normal. Take two distinct points in X, say x and y. For every $a \in x \setminus y$, there is $b \in y$ such that $a \sqcap b = \mathbf{0}$ (otherwise y would not be maximal). By normality, there are $c, d \in L$ such that $d \sqcap a = \mathbf{0}$, $c \sqcap b = \mathbf{0}$ and $c \sqcup d = \mathbf{1}$. Obviously, $a \leq c$ and $b \leq d$. Consider open neighbourhoods $O_1 = X \setminus U(c)$ of y and $O_2 = X \setminus U(d)$ of x. We show that $O_1 \cap O_2 = \emptyset$. Suppose that there is an ultrafilter u missing both c and d, in other words, there exists $j \in u$ such that $j \sqcap c = \mathbf{0} = j \sqcap d$. From distributivity, $j = j \sqcap (c \sqcup d) = (j \sqcap c) \sqcup (j \sqcap d) = \mathbf{0}$, which is a contradiction. Thus L being normal implies X being Hausdorff.

For the reverse implication, suppose that the space X is compact Hausdorff, hence normal. For any two elements $a, b \in L$ with $a \sqcap b = \mathbf{0}$, take the corresponding closed subsets $U(a)$ and $U(b)$ of X. As $U(a) \cap U(b) = U(a \sqcap b) = \emptyset$, we can find open sets O and V from the open base $\{X \setminus A \mid A \in L\}$ such that $U(a) \subset O$, $U(b) \subset V$ and $O \cap V = \emptyset$. Then $X \setminus O$ and $X \setminus V$ belong to L and satisty $U(a) \cap X \setminus O = \emptyset$, $U(b) \cap X \setminus V = \emptyset$ and $X \setminus V \cup X \setminus O = X$. We can conclude that L is normal. $\square$

To emphasize that a space is the Wallman representation for a distributive lattice L, we will denote it wL. When the lattice L is isomorphic to a lattice base for closed subsets in wL, we use the same letter for an element of L and the corresponding closed subset of wL.

Given a T_1-space X and L any lattice base for X, then the Wallman's representation wL of L is a compact space that contains X as its dense subset. We call X^* the *Wallman-Čech compactification* of X. If X is normal, then wL is equal to the Čech-Stone compactification $\beta(X)$ of X.

Remark 2.2.2. If X is a compact Hausdorff space and 2^X the lattice of all its closed subsets, then the Wallman's representation of 2^X is again X (because of the finite intersection property of 2^X, all the ultrafilters on 2^X are principal). Even more is true: If B is a lattice base for X then $wB = X$: the map $x \mapsto \{b \in B \mid x \in b\}$ is a homeomorphism. The reason is that for every $a, b \in 2^X$ with $a \sqcap b = \mathbf{0}$ there are $c, d \in B$ such that $a \leq c$, $b \leq d$ and $c \sqcap d = \mathbf{0}$.

Let $\mathcal{F}$ be a proper filter on a distributive lattice L. In the set of all filters containing $\mathcal{F}$ we can introduce lattice operations as follows:

$$\begin{aligned} \mathcal{F}_1 \sqcap \mathcal{F}_2 &= \{a \sqcap b \mid a \in \mathcal{F}_1, b \in \mathcal{F}_2\} \\ \mathcal{F}_1 \sqcup \mathcal{F}_2 &= \{a \sqcup b \mid a \in \mathcal{F}_1, b \in \mathcal{F}_2\}. \end{aligned}$$

Then the map $a \mapsto$ (the filter generated by $\{a\} \cup \mathcal{F}$) is a lattice homomorphism. Let us denote the image of L under this homomorphism by $L/\mathcal{F}$.

Proposition 2.2.3. *$w(L/\mathcal{F})$ is homeomorphic to the subspace of $w(L)$ consisting of the ultrafilters on L extending $\mathcal{F}$.*

Wallman's representation can be extended to homomorphisms between normal distributive disjunctive lattices. Indeed, let $f : A \to B$ be such a homomorphism. For $q \in wB$, define $w(f)(q) = \{a \in A \mid f(a) \in q\}$. Then $w(f)(q)$ is a proper prime filter on A, which has a unique extension to an ultrafilter (thanks to normality). So we can think of $w(f)(q)$ as of a point in wA. As $w(f)^{-1}[a] = f(a)$ the map $w(f)$ is continuous. If f is one-to-one, then $w(f)$ is onto.

It is easy to verify that the Wallman's representation gives us a functor w from the category of normal distributive disjunctive lattices and lattice-homomorphisms to the category of compact Hausdorff spaces and continuous functions.

2.3 Elementarity and ultra(co)products

Fix a first-order language $\mathcal{L}$. We denote both an $\mathcal{L}$-structure and its underlying set by the same letter and the cardinality of the underlying set by $|\cdot|$. For an extensive study on model theory see for instance books [6] or [15].

Definition 2.3.1. Let A and B be two structures for the language $\mathcal{L}$. We say that B is an *elementary substructure* of A (denoted by $B \prec A$) if B is a substructure of A and for every formula $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots a_n \in B$

$$B \models \phi[a_1, \ldots, a_n] \text{ if and only if } A \models \phi[a_1, \ldots, a_n].$$

The following result provides a useful tool to verify whether an $\mathcal{L}$-structure is an elementary substructure of a given $\mathcal{L}$-structure.

Theorem 2.3.2 (Tarski's test)**.** *Let A and B be two $\mathcal{L}$-structures with $B \subset A$. B is an elementary substructure of A if and only if for every formula $\phi(x_1, \ldots, x_n, y)$ and any n-tuple $(a_1, \ldots, a_n)$ from B if $A \models \exists y\ \phi[a_1, \ldots, a_n]$ then there is $b \in B$ such that $A \models \phi[a_1, \ldots, a_n, b]$.*

The famous Löwenheim-Skolem theorem from the beginning of the 20th century states that there are many elementary substructures for every infinite structure. The proof can be found for instance in [6].

Theorem 2.3.3 (Löwenheim-Skolem)**.** *Let A be an infinite $\mathcal{L}$-structure and let $X \subset A$. Denote $\kappa = max(|\mathcal{L}|, |X|)$. Then for every cardinal λ such that $\kappa \le \lambda \le |A|$, there exists an elementary substructure B of A such that $X \subset B$ and $|B| = \lambda$.*

Definition 2.3.4 ((Ultra)filter)**.** By a *filter* on a non-empty set I we mean a subset $\mathcal{U}$ of the power set of I such that

(i) $I \in \mathcal{U}$,

(ii) if $F, G \in \mathcal{U}$ then $F \cap G \in \mathcal{U}$,

(iii) if $F \subset G \subset I$ and $F \in \mathcal{U}$ then $G \in \mathcal{U}$.

The filter $\mathcal{U}$ is called an *ultrafilter* if in addition it holds

(iv) $F \in \mathcal{U}$ if and only if $I \setminus F \notin \mathcal{U}$.

Remark 2.3.5. In the section on lattices, we defined an ultrafilter to be a maximal proper filter on a lattice. That definition agrees with the one given above, since the ultrafilters on a set are exactly the maximal proper filters on the lattice of all subsets of I.

We use an ultrafilter on a set to construct an $\mathcal{L}$-structure (called ultraproduct) from a given family of $\mathcal{L}$-structures, that satisfies a first-order formula if and only if ultrafilter many members of the family satisfy the formula.

Definition 2.3.6 (Ultraproduct)**.** Let $\langle A_i \mid i \in I\rangle$ be a family of $\mathcal{L}$-structures with I a non-empty set and let $\mathcal{U}$ be an ultrafilter on the set I. The *ultraproduct* of $\langle A_i \mid i \in I\rangle$ is an $\mathcal{L}$-structure denoted by $\prod_{\mathcal{U}} A_i$ with the underlying set the Cartesian product $\prod_{i\in I} A_i$ in which we identify two functions f and g ($f \sim g$) whenever $\{i \in I \mid f(i) = g(i)\} \in \mathcal{U}$. It is easy to see that '$\sim$' is an equivalence relation. We denote the equivalence class of the function f by $[f]_{\mathcal{U}}$.

The interpretation of the signature of $\mathcal{L}$ in $\prod_{\mathcal{U}} A_i$ is as follows.

If G_i is the interpretation of a function symbol F of $\mathcal{L}$ in A_i, define the interpretation G of F in $\prod_{\mathcal{U}} A_i$ by $G([f_1]_{\mathcal{U}}, \dots, [f_n]_{\mathcal{U}}) = [f]_{\mathcal{U}}$ if and only if $\{i \in I \mid G_i(f_1(i), \dots, f_n(i)) = f(i)\} \in \mathcal{U}$.

If R_i is the interpretation of a relation symbol P of $\mathcal{L}$ in A_i, define the interpretation R of P in $\prod_{\mathcal{U}} A_i$ by $R([f_1]_{\mathcal{U}}, \dots, [f_m]_{\mathcal{U}})$ if and only if $\{i \in I \mid R_i(f_1(i), \dots, f_m(i))\} \in \mathcal{U}$.

If d_i is the interpretation of a constant c of $\mathcal{L}$ in A_i, define the interpretation $[d]_{\mathcal{U}}$ of c in $\prod_{\mathcal{U}} A_i$ as the class of the function $d(i) = c_i$.

If all the structures in the family are equal to some $\mathcal{L}$-structure A, we call their ultraproduct an *ultrapower* and denote $\prod_{\mathcal{U}} A$.

For correctness of the definition, see for instance [6].

Theorem 2.3.7 (The Fundamental Theorem of Ultraproducts)**.** *For any $\mathcal{L}$-formula $\phi(x_1, \dots, x_n)$ and any n-tuple $[f_1]_{\mathcal{U}}, \dots, [f_n]_{\mathcal{U}}$ from $\prod_{\mathcal{U}} A_i$,*

$$\prod_{\mathcal{U}} A_i \models \phi[[f_1]_{\mathcal{U}}, \dots, [f_n]_{\mathcal{U}}] \text{ if and only if } \{i \in I \mid A_i \models \phi[f_1(i), \dots, f_n(i)]\} \in \mathcal{U}.$$

Idea of the proof. The proof goes by induction on the complexity of the formula ϕ. The base step of the induction is when ϕ is an atomic formula. The complete proof is given for example in [6]. □

Corollary 2.3.8. *The diagonal embedding of an $\mathcal{L}$-structure A into its ultrapower $\prod_{\mathcal{U}} A$, sending a point $a \in A$ to $[(a)_i]_{\mathcal{U}}$, is an elementary embedding.*

The following powerful theorem saying that every two elementarily equivalent models have isomorphic ultrapowers is due to Shelah ([29]). It generalizes Keisler ([17]) who assumed the general continuum hypothesis. We will need the theorem in its full power in the proof of the reflection of span zero.

Theorem 2.3.9 (Shelah)**.** *Let κ be a cardinal, $\lambda = min\{\mu \mid \kappa^\mu > \kappa\}$ and let A and B be two elementarily equivalent $\mathcal{L}$-structures with $card(A), card(B) < \lambda$. Then there exists an ultrafilter $\mathcal{U}$ over κ such that $\prod_{\mathcal{U}} A$ and $\prod_{\mathcal{U}} B$ are isomorphic.*

Definition 2.3.10 (Topological ultraproduct)**.** Let $\langle X_i \mid i \in I\rangle$ be a family of topological spaces, I a non-empty index set and $\mathcal{U}$ an ultrafilter on I. The topological ultraproduct $\prod_{\mathcal{U}} X_i$ is the topological space whose points are the elements of the ultraproduct of the underlying sets and the topology is generated by "open ultraboxes" $\prod_{\mathcal{U}} O_i$, where each O_i is open in X_i. If all the spaces in the family are the same, we call their topological ultraproduct a *topological ultrapower*

We can define an ultraproduct in the purely categorial language as follows:

Definition 2.3.11 (Categorial ultraproduct)**.** Let $\mathcal{C}$ be a category, let $\langle C_i \mid i \in I\rangle$ be a family of $\mathcal{C}$-objects and let $\mathcal{U}$ be an ultrafilter on I. Consider the diagram consisting of objects of the form $\prod_{i\in J} C_i$ for $J \in \mathcal{U}$ and morphisms being the natural projections $\prod_{i\in J} C_i \to \prod_{i\in K} C_i$ for $J, K \in \mathcal{U}$, $J \supset K$. We call the direct limit of this diagram a *categorial ultraproduct* of the family $\langle C_i \mid i \in I\rangle$.

For the category of $\mathcal{L}$-structures, this definition agrees with the one given before.

However, in the category of topological spaces, the categorial ultraproduct is an indiscrete space whenever the given ultrafilter is countably incomplete. So the topological ultracoproduct usually fails to be equal to the categorial one.

On the other hand, categorial dualization of the ultraproduct notion, so called ultracoproduct, turned out to be a useful tool in the study of topological spaces.

Definition 2.3.12 (Ultracoproduct)**.** Let $\mathcal{C}$ be a category, let $\langle C_i \mid i \in I\rangle$ be a family of $\mathcal{C}$-objects and let $\mathcal{U}$ be an ultrafilter on I. Consider the diagram consisting of objects of the form $\coprod_{i\in J} C_i$ for $J \in \mathcal{U}$ and morphisms being the natural morphisms $\coprod_{i\in J} C_i \to \coprod_{i\in K} C_i$ for $J, K \in \mathcal{U},\ J \subset K$. We call the inverse limit of this diagram a *ultracoproduct* of the family $\langle C_i \mid i \in I\rangle$ and denote it by $\sum_{\mathcal{U}} C_i$.

The Stone duality shows that in the category of Stone spaces, the ultracoproduct is the Stone space of the Boolean algebra which is the ultraproduct of Boolean algebras of the spaces in the diagram.

The next two lemmas give concrete descriptions of ultracoproduct in the category of compact Hausdorff spaces, first by purely topological means and the other one by Wallman's represenation theorem.

For a Tychonoff space X let $\beta(X)$ denote the Čech-Stone compactification of X.

Lemma 2.3.13. *Let $\langle X_i \mid i \in I\rangle$ be a family of compact Hausdorff spaces with I a non-empty index set and let $\mathcal{U}$ be an ultrafilter on I. Denote $q : \bigcup_{i\in I} X_i \times \{i\} \to I$ the map sending each point to its second coordinate with I taken as a discrete space. Let $\beta(q) : \beta(\bigcup_{i\in I} X_i \times \{i\}) \to \beta(I)$ be the Čech-Stone lifting of q. Then the ultracoproduct of $\langle X_i \mid i \in I\rangle$ with respect to $\mathcal{U}$ is homeomorphic to $(\beta(q))^{-1}[\mathcal{U}]$.*

Proof. Follows from
$$\coprod_{\mathcal{U}} X_i = \beta(\bigcup_{i\in I} X_i \times \{i\}).$$
□

Lemma 2.3.14. *Let $\langle X_i \mid i \in I\rangle$ be a family of compact Hausdorff spaces with I a non-empty index set and let $\mathcal{U}$ be an ultrafilter on I. Let B_i be a lattice base for X_i. Then the ultracoproduct of $\langle X_i \mid i \in I\rangle$ with respect to $\mathcal{U}$ is homeomorphic to $w(\prod_{\mathcal{U}} B_i)$.*

Proof. We saw in Section 2.2, that we can assume that B_i is the lattice of all closed subsets of X_i for every i. The ultraproduct $\prod_{i\in I} B_i$ forms a lattice base for $\bigcup_{i\in I} X_i \times \{i\}$, so $w(\prod_{i\in I} B_i) = \beta(\bigcup_{i\in I} X_i \times \{i\})$. We can regard $\mathcal{U}$ as a filter on $\prod_{i\in I} B_i$ consisting of the points $(b_i)_{i\in I} \in \prod_{i\in I} B_i$ such that $\{i \in I \mid b_i = X_i\} \in \mathcal{U}$. Then $\prod_{\mathcal{U}} B_i$ is isomorphic to $\prod_{i\in I} B_i/\mathcal{U}$ defined in Section 2.2. Proposition 2.2.3 finishes the proof. □

As $\prod_{\mathcal{U}} B_i$ is obviously a lattice base for the topological ultraproduct $\prod_{\mathcal{U}} X_i$, we get the following corollary.

Corollary 2.3.15. $\sum_{\mathcal{U}} X_i$ *is the Wallman-Čech compactification of* $\prod_{\mathcal{U}} X_i$.

Next definition is a natural dualization of the diagonal embedding.

Definition 2.3.16 (Codiagonal map)**.** Let X be a compact Hausdorff space and $\mathcal{U}$ an ultrafilter on I. The *codiagonal map* $\nabla : \sum_{\mathcal{U}} X \to X$ sends a point p to x if and only if for each open neighbourhood U of x the ultrapower $\prod_{\mathcal{U}} U$ contains a member of p.

A topological and a Wallman-type description of the codiagonal map follow.

Lemma 2.3.17. *Let* $\beta(p) : \beta(X \times I) \to \beta(X)$ *be the Čech-Stone lifting of the map* $p : X \times I \to X$ *sending each point to its first coordinate. Then the codiagonal map is the restriction of* $\beta(p)$ *to the ultracoproduct.*

Lemma 2.3.18. *Let* $\Delta : A \to \prod_{\mathcal{U}} A$ *be the diagonal embedding of a distributive disjunctive normal lattice* A *to its ultrapower. Then* $w(\Delta) = \nabla$.

Proposition 2.3.19. *The clopen algebra of the ultracoproduct* $\sum_{\mathcal{U}} X_i$ *is isomorphic with the ultracoproduct of the clopen algebras of* X_i*'s. In particular,* $\sum_{\mathcal{U}} X_i$ *is connected if and only if* $\{i \mid X_i \text{ is connected }\} \in \mathcal{U}$.

Definition 2.3.20. Let $\phi_i : X_i \to Y_i$ for $i \in I$ be continuous mappings between continua and let $\mathcal{U}$ be an ultrafilter on I. Let ϕ be the sum of the maps $\langle \phi_i \mid i \in I \rangle$ and $\beta(\phi)$ the Čech-Stone lifting of ϕ. The *ultracoproduct of maps* $\langle \phi_i \mid i \in I \rangle$ (denoted by $\sum_{\mathcal{U}} \phi_i$) is the restriction of $\beta(\phi)$ to the ultracoproduct $\sum_{\mathcal{U}} X_i$.

2.4 Elementary sublattices

Elementarity turned out to be a useful tool in the theory of compact Hausdorff spaces. If X is a compact Hausdorff space and 2^X the lattice of all its closed subsets, the Wallman's representation of any elemetary sublattice of L of 2^X is also compact Hausdorff, because being normal, distributive and disjunctive are expressible by a first-order formula.

Existence of elementary sublattices of any infinite cardinality up to the cardinality of the original lattice is given by the Löwenheim-Skolem theorem. We often want to convert a non-metric compact space X to a metric one. That can be achieved by taking elementary sublattices of 2^X of countable cardinality:

Theorem 2.4.1. *Let L be a normal distributive disjunctive lattice of countable cardinality. Then wL is compact, Hausdorff and metrizable.*

Proof. From Wallman's representation theorem we know that wL is compact Hausdorff, hence normal, and that L is isomorphic to a base for closed set in wL. By Urysohn's metrization theorem wL is metrizable. $\square$

We are interested in what properties of compact Hausdorff spaces are inherited by the Wallman's representations of all elementary sublattices of the lattice of all closed sets. We call such properties *elementarily reflected.*

Sometimes it is useful to know more about the structure of the elementary sublattice. We can accomplish that by using models for almost all of set theory to get elementary sublattices. Recall that for a cardinal θ, $H(\theta)$ denotes the set of all sets whose transitive closure has cardinality less then θ. These sets are very important and useful because if θ is uncountable regular then

$$H(\theta) \models \text{ ZFC - P } \text{ (see Chapter IV in [18])} .$$

When proving a property of some object, the arguments use only a limited number of sets. So there is θ large enough to guarantee that $H(\theta)$ contains all sets needed (for example, see Levy Reflection Theorem, Chapter IV in [18]). In the main chapter we choose a cardinal θ large enough so that $H(\theta)$ contains all sets necessary to investigate the properties we deal with.

Löwenheim-Skolem theorem now provides elementary submodels of $H(\theta)$. If we take an elementary submodel $\mathcal{M}$ of $H(\theta)$ that contains 2^X as its element, then $\mathcal{M} \cap 2^X$ is an elementary sublattice of 2^X that contains all elements of 2^X expressible by a formula in the language of set theory with coefficients in $\mathcal{M}$ and we can see more of its behaviour from outside within the frame of the model $\mathcal{M}$. Thus the Wallman's representation of $\mathcal{M} \cap 2^X$ shares more properties with the compactum X. The properties that are inherited from X by the Wallman's representation of $\mathcal{M} \cap 2^X$ for any $2^X \in \mathcal{M} \prec H(\theta)$ are called *elementarily reflected by submodels.*

An important class of properties that are elementarily reflected are those that do not depend on the choice of the lattice base. This leads to the notion of a base-free formula that is due to Bankston (for instance [5]). A lattice formula ϕ is called *base-free* if for any compact space X and any lattice base $\mathcal{B}$ for closed sets of X

$$\mathcal{B} \models \phi \text{ if and only if } 2^X \models \phi.$$

For example connectedness, (hereditarily) indecomposability and having covering dimension less or equal to $n \in \omega$ are properties that can be expressed by a base-free formula ([5]).

As we are particulary interested in continua, we give two proofs that being connected is elementarily reflected. First showing that connectedness is expressible by a base-free formula and then deriving some properties of $w(e)$, the Wallman's representation of the elementary embedding e of an elementary sublattice L of 2^X into 2^X.

Let $\mathrm{conn}(z)$ denote the following formula

$$\mathrm{conn}(z) = \forall xy\ (x \sqcup y = z \wedge x \sqcap y = \mathbf{0} \rightarrow x \sqcap z = \mathbf{0} \vee y \sqcap z = \mathbf{0}).$$

So for compact space X and $a \in 2^X$, $\mathrm{conn}[a]$ is true if and only if a is a connected subset of X.

To prove that $\mathrm{conn}[\mathbf{1}]$ is base-free, we will need the following lemma.

Lemma 2.4.2. *Let X be a compact Hausdorff space and $\mathcal{B}$ a lattice base for X. If $F \subset O$, where F is a closed and O an open subset of X then there is a $B \in \mathcal{B}$ such that $F \subset B \subset O$.*

Theorem 2.4.3. *The lattice sentence conn$[\mathbf{1}]$ expressing the connectedness is base-free.*

Proof. Let X be a compact Hausdorff space and $\mathcal{B}$ a lattice base for X. Obviously

$$2^X \models \mathrm{conn}[\mathbf{1}] \text{ implies } \mathcal{B} \models \mathrm{conn}[\mathbf{1}].$$

For the converse, suppose that $\mathcal{B} \models \mathrm{conn}[\mathbf{1}]$. If X is not connected, then Lemma 2.4.2 provides F and G from $\mathcal{B}$ witnessing disconectivity of X. It means that

$$\mathcal{B} \models \exists FG\ (F \neq \mathbf{0}\ \wedge\ G \neq \mathbf{0}\ \wedge\ F \sqcup G = \mathbf{1}\ \wedge\ F \sqcap G = \mathbf{0}),$$

which contradicts $\mathcal{B}$ modelling $\mathrm{conn}[\mathbf{1}]$. □

Let L be an elementary sublattice of 2^X and let $e : L \rightarrow 2^X$ denote the elementary embedding. Applying the functor w, we get the continuous mapping $w(e) : X \rightarrow wL$ sending a point x to the ultrafilter $\{A \in L \mid x \in A\}$. As e is one-to-one, $w(e)$ is obviously onto. Combining that with X being compact and wL Hausdorff, $w(e)$ is closed. Lemma 2.8 in [4] shows that $w(e)$

is even weakly confluent (a map between two continua A and B is *weakly confluent* if for every subcontinuum C of B there is a subcontinuum of A that maps onto C). Hence many properties of compact spaces are elementarily reflected.

Theorem 2.4.4. *Being connected is a property of compact spaces that is elementarily reflected.*

3 Chainability, span and their reflection

In this chapter, we introduce two important properties for continua, chainability and span. In [19], Lelek showed that every chainable continuum has span zero and later he conjectured the converse. We give the proof of his result for arbitrary continua ([30]) and show that his conjecture can be generalized to non-metric spaces. Namely, we show that if there is a non-metric counterexample to Lelek's conjecture, then, using elementarity, we find a metric one.

3.1 Chainability

Definition 3.1.1 (Chain)**.** Let X be a continuum. A *chain* is a nonempty, finite collection $\mathcal{C} = \{C_1, \ldots, C_n\}$ of open subsets C_i of X such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$. The elements C_i of $\mathcal{C}$ are called *links* of the chain $\mathcal{C}$.

Definition 3.1.2 (Chainability)**.** A continuum X is *chainable* if every open cover has an open cover refinement which is a chain.

Remark 3.1.3. The definition above agrees with the definition of chainable for metric continua which states that a metric continuum X is chainable if for every $\varepsilon > 0$ there is a chain covering X such that all of its elements have the diameter less than ε. Chainable metric continua are sometimes called *arc-like*, as these two properties are known to be equivalent (see for instance [24]).

We prove that chainability can be rephrased in terms of a lattice base only. For that, we need a well-known theorem on shrinkings. A *shrinking* of a cover $\{O_i\}_{i\in I}$ of a space X is a cover $\{U_i\}_{i\in I}$ of the space X such that $U_i \subset O_i$ for every i.

Theorem 3.1.4. *([12]) Every finite open covering $\{V_i\}_{i=1}^k$ of a normal space X has shrinkings $\{F_i\}_{i=1}^k$ and $\{W_i\}_{i=1}^k$ of functionally closed and functionally open sets respectively such that $F_i \subset W_i \subset \overline{W}_i \subset V_i$ for all $i = 1, \dots, k$.*

Theorem 3.1.5. *Let $\mathcal{B}$ be an open base for a continuum X closed under finite unions and finite intersections. Then X is chainable if and only if every open cover consisting of sets from $\mathcal{B}$ has a chain refinement in $\mathcal{B}$.*

Proof. Take $\mathcal{V} = \{V_i\}_{i=1}^k \subset \mathcal{B}$ an open cover of X. If X is chainable, then there is an open chain refinement $\mathcal{C} = \{C_i\}_{i=1}^n$ of $\mathcal{V}$. By 3.1.4 there is a shrinking $\{D_i\}_{i=1}^n$ of $\mathcal{C}$ consisting of functionally open sets such that $\overline{D_i} \subset C_i$ for all i. By connectedness the family $\{D_i\}_{i=1}^n$ is still a chain. By compactness (and closure under finite unions) there are B_i in $\mathcal{B}$ such that $\overline{D_i} \subset B_i \subset \overline{B_i} \subset C_i$ for every $i \leq n$. □

To prove that chainability is elementarily reflected, it would be handy if it was expressible in the lattice language. There we are only partially successful, since the following formula expressing chainability uses a disjunction over natural numbers, which leads to need of the logic $\mathcal{L}_{\omega_1\omega}$.

Let $\phi_m(u_1, \dots, u_m)$ be the following formula in the lattice language in the logic $\mathcal{L}_{\omega_1\omega}$.

$$u_1 \sqcap \ldots \sqcap u_m = \mathbf{0} \to \bigvee_n \exists v_1 \ldots v_n \theta(u_1, \dots, u_m; v_1, \dots, v_n), \tag{3.1}$$

where $\theta(u_1, \dots, u_m; v_1, \dots, v_n)$ is the conjunction of the following formulas

(i) $v_1 \sqcap \ldots \sqcap v_n = \mathbf{0}$

(ii) $\bigwedge_i \bigvee_j v_i \geq u_j$

(iii) $\bigwedge_{i=1}^{n-1} v_i \sqcup v_{i+1} \neq \mathbf{1} \wedge \bigwedge_{i=1}^{n-2} \bigwedge_{j=i+2}^{n} v_i \sqcup v_j = \mathbf{1}$.

Translated into open covers, for a finite open cover $\{wL \setminus u_i \mid i < m\}$ there is a finite open cover $\{wL \setminus v_i \mid i < n\}$ (by item (i)) refining $\{wL \setminus u_i \mid i < m\}$ (by item (ii)). Finally, the formula (iii) expresses that $\{wL \setminus v_i \mid i < n\}$ forms a chain. Applying Lemma 3.1.5, we get the following

Lemma 3.1.6. *Let L be a normal distributive disjunctive lattice. Its Wallman's representation wL is chainable if and only if*

$$L \models \forall u_1 \ldots u_m \; \phi_m(u_1, \dots, u_m) \text{ for all } m < \omega.$$

Remark 3.1.7. As the unit interval $[0,1]$ is chainable but its ultracopower over ω is not chainable (see [14]), chainability is not a first-order property in the language of lattices.

However, in the language of the set theory, $\{\in\}$, the formula 3.1 becomes a first-order formula

$$u_1 \sqcap \ldots \sqcap u_m = \mathbf{0} \rightarrow (\exists n \in \omega)(\exists v \in L^n)\theta(u_1, \ldots, u_m; v).$$

We rely on that in the proofs of the reflection of chainability and non-chainability by submodels (if $\mathcal{M}$ is an elementary submodel of $H(\theta)$, then $\omega \in \mathcal{M}$).

3.2 Span

In [19], Lelek introduced the notion of span for metric spaces. Let (X, d) be a metric space. We shall denote by π_1 and π_2 the projections of the square $X \times X$ onto its axes. The *span* σX of the space X is defined to be the supremum of numbers ε for which there is a connected subset Z of $X \times X$ such that

(i) $\pi_1[Z] = \pi_2[Z]$

(ii) $d(x, y) \geq \varepsilon$ for $(x, y) \in Z$

If we drop the metric from the definition, we can only distinguish between span zero and non-zero. As span turned out to be particularly useful in the continuum theory, we give the following definition.

Definition 3.2.1 (Span)**.** A continuum X has *span* zero if every subcontinuum Z of $X \times X$, which projects onto the same set on both coordinates, has a nonempty intersection with the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$ of X. Otherwise we say that X has span non-zero.

In [21], Lelek defined variations of span, so called surjective span and surjective semispan, by changing the conditinions on the projections of the subcontinuum Z of $X \times X$ in the definition of span to $\pi_1[Z] = \pi_2[Z] = X$ and $\pi_1[Z] \subset \pi_2[Z] = X$ respectively.

Relaxing the metric from the definition, we obtain the following definition.

Definition 3.2.2. A continuum X has the *surjective (semi)span* zero if every subcontinuum Z of $X \times X$, which projects onto X in both coordinates (at least in one coordinate), has a nonempty intersection with the diagonal Δ_X of X. Otherwise we say that X has surjective (semi)span non-zero.

Lelek showed that span and surjective span need not be equal ([21]) and neither span and semispan do ([22]). However, Davis ([8]) proved that having span zero and having semispan zero are equivalent for metric continua. For his proof he defined a new kind of span, the symmetric span.

We call a subset A of the square $X \times X$ *symmetric* if $A = A^{-1}$, where $A^{-1} = \{(x, y) \mid (y, x) \in A\}$.

Definition 3.2.3 (Symmetric span)**.** Let X be a continuum. X has the *symmetric span* zero if for every symmetric subcontinuum Z of the square $X \times X$ the intersection of Z with the diagonal Δ_X of X is non-empty. Otherwise X has the symmetric span non-zero.

Theorem 3.2.4. *Every chainable continuum has span zero.*

Proof. Let X be a chainable continuum with non-zero span. Let Z be a subcontinuum of $X \times X$ that misses the diagonal Δ_X. There exists an open set $U \subset X \times X$ such that $\Delta_X \subset U$ and $U \cap Z = \emptyset$. So for every $x \in X$ we can choose an open subset U_x of X such that $U_x \times U_x \cap Z = \emptyset$. As X is chainable, the open cover $\{U_x \mid x \in X\}$ has a finite chain refinement $\{V_1, \ldots, V_n\}$. Define two open subsets U_1 and U_2 of $X \times X$ as follows:

$$U_1 = \bigcup_{i,j<n,\ i<j} V_i \times V_j$$

$$U_2 = \bigcup_{i,j<n,\ i>j} V_i \times V_j.$$

Then $U_1 \cup U_2 \supset Z$ and $U_1 \cap U_2 \cap Z = \emptyset$. If not, Z would intersect $V_i \times V_i$ for some i, which contradicts V_i being a subset of some U_x. Thus Z is the union of two disjoint nonempty open (in Z) sets $U_1 \cap Z$ and $U_2 \cap Z$. By connectedness of Z, one of these sets must be empty, say $U_2 \cap Z$. So $\pi_2[Z] \subset \pi_2[U_1] \neq X$, which means that surjective span is zero. As chainability is hereditary, it implies that span is zero as well. Since $Z \subset U_1$ and Z misses all $V_i \times V_i$, Z cannot be symmetric and hence also symmetric span must be zero. $\square$

3.3 Reflection

Let θ be a cardinal large enough so that $H(\theta)$ contains all sets needed in the reasonings below (see the section on Elementary sublattices).

For a continuum X, let 2^X (resp. $2^{X\times X}$) denote the lattice of all closed subset of X (resp. $X \times X$). Let $\mathcal{M}$ be an elementary submodel of $H(\theta)$ that contains 2^X as one of its elements. As the following sets are expressible by a set-theoretical formula with coefficients in $\mathcal{M}$, they are elements of $\mathcal{M}$.

X, $X \times X$, $2^{X\times X}$, $\pi_i : X \times X \to X$, the diagonal Δ_X of X, $\emptyset$, all natural numbers, the set of all natural numbers ω.

Hence $L = \mathcal{M} \cap 2^X$ and $K = \mathcal{M} \cap 2^{X\times X}$ are elementary sublattices of 2^X and $2^{X\times X}$ respectively.

In this section we present the proofs of the reflection of chainability, non-chainability and span non-zero by submodels from the dissertation [30]. We prove that also having span zero is elementarily reflected by submodels. Thus we have a substantially larger class of spaces to look for a counterexample to Lelek's conjecture. Indeed, if there exists a non-metric continuum X that has span zero but is not chainable, then taking $\mathcal{M}$ to be countable, the Wallman representation of $L = \mathcal{M} \cap 2^X$ will be a metric continuum with span zero, that is not chainable.

Theorem 3.3.1. *wK and $wL \times wL$ are homeomorphic.*

Proof. Define the map $\iota : wK \to wL \times wL$, by

$$\iota(p) = (\{\pi_1[A] \mid A \in p\}, \{\pi_2[A] \mid A \in p\}).$$

Claim 1. *$\iota(p) \in wL \times wL$ for all $p \in wK$.*

Proof. By elementarity, $\pi_1[A] \in L$, whenever $A \in K$, and $B \times X \in K$, whenever $B \in L$. Since p has the finite intersection property, so does $\{\pi_1[A] \mid A \in p\}$. Take any $U \in L$ such that $U \cap \pi_1[A] \neq \emptyset$ for all $A \in p$. Then $\pi_1^{-1}[U] = U \times X$ intersects all elements of p and lies in K. Since p is maximal, $U \times X$ must be an element of p, so $\{\pi_1[A] \mid A \in p\}$ is an ultrafilter on L and hence a point of wL. Analogously, $\{\pi_2[A] \mid A \in p\} \in wL$. We proved that $\mathrm{Im}(\iota) \subset wL \times wL$. $\square$

Claim 2. *ι is surjective.*

Proof. Let $(q, r) \in wL \times wL$. Thanks to elementarity, the set $\mathcal{S} = \{A \times X, X \times B \mid A \in q, B \in r\}$ is a subset of K. Since q and r have the finite intersection property and $(A \times X) \cap (X \times B) = A \times B \in K$, $\mathcal{S}$ has the finite intersection property as well. Thus there exists a point $p \in wK$ extending $\mathcal{S}$. Previous claim showed $\{\pi_1[A] \mid A \in p\} \in wL$. But also, $\{\pi_1[A] \mid A \in p\} \supset q$. By maximality of q, $\{\pi_1[A] \mid A \in p\} = q$. Similarly, $\{\pi_2[A] \mid A \in p\} = r$. Thus $\iota(p) = (q, r)$ and $\mathrm{Im}(\iota) = wL \times wL$. ☐

Claim 3. *ι is injective.*

Proof. Suppose that there are $p \neq s \in wK$ with $\iota(p) = \iota(s)$. As $p \neq s$, there are $A \in p$ and $B \in s$ such that $A \cap B = \emptyset$. By elementarity, there are open disjoint subsets U and V containing A and B respectively with disjoint closures in $\mathcal{M}$. For every point in A there is a pair of open subsets of X such that their product contains the point and its closure is contained in the closure of U. As A is compact, we can find finitely many of these pairs such that their products cover A. Since $\omega \in \mathcal{M}$, by elementarity we can find a finite number of pairs of elements of L the products of which cover A and a finite number of elements of L whose products cover B, with their unions disjoint. Since K contains all the products of pairs of elements from L and p and s are maximal filters, p and s contain a product of elements of L with empty intersection, which contradicts $\iota(p) = \iota(s)$. ☐

Claim 4. *ι is continuous.*

Proof. $\{A \times B \mid A, B \in L\}$ is a closed base for $wL \times wL$. $\iota^{-1}[A \times B] = A \times X \cap X \times B$. Since $(A \times X) \cap (X \times B) \in K$, it corresponds to a closed subset of wK. Thus ι is continuous. ☐

We showed that ι is a continuous one-to-one onto map. wK being compact and $wL \times wL$ Hausdorff implies that ι is a homeomorphism. ☐

Corollary 3.3.2. *Let $e : L \to 2^X$ and $f : K \to 2^{X \times X}$ be the elementary embeddings. Then $\iota \circ w(f) = w(e) \times w(e)$.*

Proof. For every $(x, y) \in X \times X$

$$\iota \circ w(f)(x, y) = (\{\pi_1[A] \mid (x, y) \in A,\ A \in K\}, \{\pi_2[A] \mid (x, y) \in A,\ A \in K\})$$

and

$$w(e) \times w(e)(x, y) = (\{B \in L \mid x \in B\}, \{C \in L \mid y \in C\})$$

Whenever $x \in B \in L$ and $y \in C \in L$, we have $(x, y) \in A \times B \in K$. So $\{B \in L \mid x \in B\} \subset \{\pi_1[A] \mid (x, y) \in A,\ A \in K\}$ and $\{C \in L \mid y \in C\} \subset \{\pi_2[A] \mid (x, y) \in A,\ A \in K\}$. From the above we know that all these sets are ultrafilters, so we can replace '$\subset$' by '$=$'. Hence $\iota \circ w(f)(x, y) = w(e) \times w(e)(x, y)$ for all $(x, y) \in X \times X$. □

It means that whenever a subset of $wL \times wL$ misses the diagonal of wL, its preimage under $\iota \circ w(f)$ misses the diagonal of X.

Theorem 3.3.3. *Chainability is a property of continua that is elementarily reflected by submodels.*

Proof. Suppose that X is chainable. We will show that the continuum wL is chainable as well. In 3.1.5 we proved that it is enough to consider open covers of wL consisting of the elements from the base $\mathcal{B} = \{wL \setminus F \mid F \in L\}$ only. So let $\mathcal{V} = \{v_1, \dots, v_m\} \subset \mathcal{B}$ be an open cover of wL.

Since 2^X models the formula 3.1, there exists $n \in \omega$ and $w_1, \dots, w_n \in 2^X$ such that $\theta[v_1, \dots, v_k; w_1, \dots, w_n]$, where θ is defined in 3.1. By elementarity, there are $\{x_1, \dots, x_n\} \in L$ such that $\theta[v_1, \dots, v_m; x_1, \dots, x_n]$. Then $\{wL \setminus x_i \mid i = 1, \dots, n\}$ is an open chain cover of wL refining $\mathcal{V}$. □

Theorem 3.3.4. *Being non-chainable is a property of continua that is elementarily reflected by submodels.*

Proof. X being non-chainable means that there is an open cover $\{U_1, \dots, U_m\}$ every open cover refinement of which does not form a chain. Since $\omega \in \mathcal{M}$, elementarity provides $n \in \omega$ and $\{u_0, \dots, u_{n-1}\} \in \mathcal{M}$ such that

$$\mathcal{M} \models \{u_0, \dots, u_{n-1}\} \subset 2^X \wedge \bigcap_{i<n} u_i = \emptyset$$

and $\{X \setminus u_0, \dots, X \setminus u_{n-1}\}$ is an open cover of X that does not have a chain refinement. In the language of closed sets, for every $m \in \omega \subset \mathcal{M}$ and every $\{v_0, \dots, v_{m-1}\} \subset 2^X \cap \mathcal{M}$, if

$$\mathcal{M} \models \bigcap_{i<m} v_i = \emptyset \wedge \bigwedge_{i<m} \bigvee_{j<n} u_j \subset v_i \tag{3.2}$$

then also

$$\mathcal{M} \models \bigvee \{v_i \cup v_j \neq X \mid |i - j| \geq 2\} \vee \bigvee \{v_i \cap v_{i+1} = X \mid i < m - 1\}. \tag{3.3}$$

Suppose that wL is chainable. Since $\{u_1, \dots, u_n\} \subset L = 2^X \cap \mathcal{M}$, by Lemma 3.1.5 there is $m \in \omega$ and a subset $\{v_i \mid i < m\}$ of L such that their complements form a chain cover refinement of the cover $\{wL \setminus u_i \mid i < n\}$. It means that $\mathcal{M}$ models 3.2 but also $\mathcal{M}$ models the negation of 3.3, which is a contradiction. $\square$

In order to prove reflection of span non-zero and span zero, we introduce three unary functions p_1, p_1, i' on $2^{X \times X}$.

The projections π_1, π_2 induce semi-lattice homomorphisms

$$\begin{aligned} \pi_1' &: 2^{X\times X} \to 2^X \quad A \mapsto \pi_1[A] \\ \pi_2' &: 2^{X\times X} \to 2^X \quad A \mapsto \pi_2[A] \end{aligned}$$

having as right inverses lattice monomorphisms

$$\begin{aligned} e_1 &: 2^X \to 2^{X\times X} \quad A \mapsto A \times X \\ e_2 &: 2^X \to 2^{X\times X} \quad A \mapsto X \times A \end{aligned}$$

respectively. We make π_i''s into mappings from $2^{X\times X}$ to $2^{X\times X}$ by composing them with e_i's

$$\begin{aligned} p_1 &:= e_1 \circ \pi_1' : 2^{X\times X} \to 2^{X\times X} \quad A \mapsto \pi_1[A] \times X \\ p_2 &:= e_2 \circ \pi_2' : 2^{X\times X} \to 2^{X\times X} \quad A \mapsto X \times \pi_2[A]. \end{aligned}$$

Since e_i is a monomorphisms and π_i' is onto, $\mathrm{Im}(p_i)$ is lattice-isomorphic to 2^X.

To deal with symmetric span, we define a mapping

$$i : X \times X \to X \times X \quad (x, y) \mapsto (y, x),$$

which induces the lattice isomorphism

$$i' : 2^{X\times X} \to 2^{X\times X} \quad A \mapsto A^{-1}.$$

By elementarity, $\pi_1', \pi_2', e_1, e_2, p_1, p_2$ and i' are elements of $\mathcal{M}$. Let us denote the restriction of π_i' to K by $\pi_i'^K$ and the restriction of e_i to L by e_i^L.

Theorem 3.3.5. *Having span non-zero is a property of continua that is elementarily reflected by submodels.*

Proof. If X has span non-zero then there is a subcontinuum $Z \subset X \times X$ that projects onto both coordinates onto the same set and that misses the diagonal Δ_X. It implies that for every closed subset $F \in 2^X$ we have

$$Z \leq e_1[F] \text{ if and only if } Z \leq e_2[F].$$

Consider a family $\mathcal{G}$ of all closed subsets G of X such that $Z \leq e_1[G] \sqcup e_2[G]$. As $Z \sqcap \Delta_X = \mathbf{0}$, $\bigsqcap_{\mathcal{G}} G = \mathbf{0}$. By compactness there is a finite subfamily $\mathcal{G}' \subset \mathcal{G}$ having zero meet. By elementarity there is such a Z in $\mathcal{M}$; as Δ_X also belongs to $\mathcal{M}$ we get such a continuum in $wL \times wL$. $\square$

Hereinafter, we regard $2^{X \times X}$ as a lattice algebra with three additional unary functions p_1, p_2, i', so as a model for the language $\mathcal{L} = \{\sqcup, \sqcap, \mathbf{0}, \mathbf{1}, p_1, p_2, i'\}$, and K as an $\mathcal{L}$-substructure of $2^{X \times X}$. Let us denote the interpretation of a function symbol f in K by f^K.

Thanks to elementarity, $\mathrm{Im}(p_1^K)$ and $\mathrm{Im}(p_2^K)$ are lattice-isomorphic to L. From the definition of the ultraproduct, $\mathrm{Im}(\prod_{\mathcal{U}} p_i)$ is lattice-isomorphic to $\prod_{\mathcal{U}} 2^X$ and $\mathrm{Im}(\prod_{\mathcal{U}} p_i^K)$ is lattice-isomorphic to $\prod_{\mathcal{U}} L$.

Theorem 3.3.6. *Having (surjective) (semi)span zero and symmetric span zero are properties of continua that are elemetarily reflected by submodels.*

Proof. Let X be a continuum and L and K as above. Denote $e : K \to 2^{X \times X}$ the elementary embedding. After adding the elements of K to the language $\mathcal{L}$ as constants, the models K and $2^{X \times X}$ remain elementarily equivalent. By Shelah's theorem (2.3.9), there are a cardinal κ, an ultrafilter $\mathcal{U}$ on κ and an isomorphism $h : \prod_{\mathcal{U}} K \to \prod_{\mathcal{U}} 2^{X \times X}$ such that $\Delta \circ e = h \circ \Delta$, where Δ stands for the diagonal embedding of a structure into its ultrapower.

$$\begin{tikzcd}
K \arrow[r, "e"] \arrow[d, "\Delta"'] & 2^{X \times X} \arrow[d, "\Delta"] \\
\prod_{\mathcal{U}} K \arrow[r, "h"] \arrow[d, "\prod_{\mathcal{U}} p_i^K"'] & \prod_{\mathcal{U}} 2^{X \times X} \arrow[d, "\prod_{\mathcal{U}} p_i"] \\
\prod_{\mathcal{U}} K \arrow[r, "h"'] & \prod_{\mathcal{U}} 2^{X \times X}
\end{tikzcd} \tag{3.4}$$

Applying the functor w, we obtain a homeomorpism $w(h) : \sum_{\mathcal{U}} X \times X \to \sum_{\mathcal{U}} wL \times wL = w(\prod_{\mathcal{U}} K)$ such that $\nabla \circ w(h) = w(e) \circ \nabla$, where $\nabla = w(\Delta)$ is the corresponding codiagonal map.

Let $Z \subset wL \times wL$ witness (surjective)(semi)span (symmetric span) non-zero. Define $Z_X = \nabla \circ w(h)^{-1}[\sum_{\mathcal{U}} Z]$. We show that Z_X is a subcontinuum of $X \times X$ that maps onto Z (so it does not meet the diagonal) and that has the same "projection properties" as Z, so X has the same kind of span non-zero as wL.

By 2.3.19, $\sum_{\mathcal{U}} Z$ is a subcontinuum of $\sum_{\mathcal{U}} wK$ and consequently Z_X is a subcontinuum of $X \times X$.

We first show that $\nabla[\sum_{\mathcal{U}} Z] = Z$ using the description of ∇ from Lemma 2.3.17. On one hand, $\beta(p)[\sum_{\mathcal{U}} Z] \subset \beta(p)[\beta(Z \times I)] = \overline{p[Z \times I]} = Z$ (for a subset S of a space Y, $\overline{S}$ denotes the closure of S in Y). On the other hand, $Z = p[Z \times I] \subset \beta(p)[\sum_{\mathcal{U}} Z]$: in $\beta(X \times I)$ the closure of $\{z\} \times I$ intersects $\sum_{\mathcal{U}} Z$ and the points in the intersection get mapped to z.

From $\nabla[\sum_{\mathcal{U}} Z] = w(e) \circ \nabla \circ w(h)^{-1}[\sum_{\mathcal{U}} Z]$ it follows that $w(e)[Z_X] = Z$, which means that Z_X misses the diagonal of X.

In the rest of the proof, we deduce that Z_X inherits the "projection properties" (being symmetric) from Z.

Since h is an isomorphism, we have that $h \circ \prod_{\mathcal{U}} p_1^K = \prod_{\mathcal{U}} p_1 \circ h$ and similarly for p_2 and i'. As π_i' and $\pi_i'^K$ are surjective and e_i and e_i^L are injective it implies that $h \circ \prod_{\mathcal{U}} e_i^L = \prod_{\mathcal{U}} e_i \circ h^L$, where h^L denotes h restricted to the isomorphic copy of $\prod_{\mathcal{U}} L$ in $\prod_{\mathcal{U}} K$.

The maps e_1^L and e_2^L induce the first and the second projections π_1^{wK}, π_2^{wK} : $wK = wL \times wL \to wL$ respectively via the Wallman representation. Similarly $\prod_{\mathcal{U}} e_i^L$ induces $\sum_{\mathcal{U}} \pi_i^{wK} : \sum_{\mathcal{U}} wL \times wL \to \sum_{\mathcal{U}} wL$ and $\prod_{\mathcal{U}} e_i$ induces $\sum_{\mathcal{U}} \pi_i : \sum_{\mathcal{U}} X \times X \to \sum_{\mathcal{U}} X$.

As $h \circ \prod_{\mathcal{U}} e_i^L = \prod_{\mathcal{U}} e_i \circ h^L$, also $w(h^L) \circ \sum_{\mathcal{U}} \pi_i = \sum_{\mathcal{U}} \pi_i^{wK} \circ w(h)$.

Since $\nabla[\sum_{\mathcal{U}} Z] = Z$, $\pi_i[\sum_{\mathcal{U}} Z] = \sum_{\mathcal{U}} \pi_i[Z]$ and $\nabla \circ \sum_{\mathcal{U}} \pi_i^{wK} = \pi_i^{wK} \circ \nabla$,

$$\begin{aligned}
\pi_1^K[Z] &\subset \pi_2^K[Z] \text{ if and only if } \sum_{\mathcal{U}} \pi_1^K\left[\sum_{\mathcal{U}} Z\right] \subset \sum_{\mathcal{U}} \pi_2^K\left[\sum_{\mathcal{U}} Z\right] \\
\pi_1^K[Z] &= \pi_2^K[Z] \text{ if and only if } \sum_{\mathcal{U}} \pi_1^K\left[\sum_{\mathcal{U}} Z\right] = \sum_{\mathcal{U}} \pi_2^K\left[\sum_{\mathcal{U}} Z\right] \\
\pi_1^K[Z] &= wL \text{ if and only if } \sum_{\mathcal{U}} \pi_1^K\left[\sum_{\mathcal{U}} Z\right] = \sum_{\mathcal{U}} wL
\end{aligned}$$

As $\sum_{\mathcal{U}} \pi_i = w(h_L)^{-1} \circ \sum_{\mathcal{U}} \pi_i^{wK} \circ w(h)$, $w(h)^{-1}[\sum_{\mathcal{U}} Z]$ has the same relation between the projections on the axes as $\sum_{\mathcal{U}} Z$ and consequently Z_X as well.

The isomorphism i'^K obviously induces $i^K := \iota \circ w(i'^K) \circ \iota^{-1} : wL \times wL \to wL \times wL$ that switches the coordinates. Since $\prod_{\mathcal{U}} i'^K \circ h = h \circ \prod_{\mathcal{U}} i'$, it holds that $\sum_{\mathcal{U}} i^K[A] = A$ if and only if $\sum_{\mathcal{U}} i[w(h)[A]]) = w(h)[A]$ for $A \subset \sum_{\mathcal{U}} wL \times wL$. It is easy to see that $i^K[Z] = Z$ if and only if $\sum_{\mathcal{U}} i^K [\sum_{\mathcal{U}} Z] = \sum_{\mathcal{U}} Z$, so $i^K[Z] = Z$ if and only if $i[Z_X] = Z_X$.

We proved that whenever Z witnesses (surjective)(semi)span (symmetric span) non-zero, then Z_X witnesses the same kind of span non-zero in X. $\square$

4 Questions

We proved that span zero is reflected by submodels using Shelah's theorem, which is a very deep and complicated result. Thus we ask

Question 2. *Is there an easier (more direct) proof of the reflection of span zero?*

In metric case, the confluent onto mappings between continua with their square confluent preserve span zero (see [10]). Hence a natural question arises

Question 3. *Let $f : X \to Y$ be a confluent mapping from a continuum X onto a continuum Y, let span of X be zero and $f \times f$ confluent. Does Y have span zero?*

Bankston ([4]) proved that the Wallman representation of an elementary embedding is weakly confluent. It is natural to ask whether it is confluent. Although the answer is negative in general (there are codiagonal maps that are not confluent [1]), the following question is still of major interest.

Question 4. *If L is an elementary sublattice of 2^X, is the Wallman representation of the elementary embedding of L into 2^X confluent?*

Bankston used in the proof of weak confluence Shelah's theorem.

Question 5. *Is there a proof of (weak) confluence without the use of Shelah's theorem?*

We would be successful if the following question had the affirmative answer,

Question 6. *Let L be an elementary sublattice of 2^X. Can we prove that L contains a dense subset of the hyperspace of all subcontinua of wL without invoking Shelah's theorem?*

because

Theorem 4.0.7. *Let $e : L \to 2^X$ be the elementary embedding of a countable lattice L to 2^X. Then L contains a dense subset of the hyperspace of all subcontinua of wL if and only if $w(e)$ is weakly confluent.*

By the *hyperspace* $\mathcal{C}$ of a continuum Y we mean the space whose points are all subcontinua of Y and the topology is generated by all sets of the form

$$\langle U_1, \ldots, U_n \rangle = \{C \mid C \subset \bigcup_{i=1}^{n} U_i \text{ and } C \cap U_i \neq \emptyset \text{ for every } i = 1, \ldots, n\}$$

with U_i open in Y, for $i = 1, \ldots, n$. It is easy to verify that the open sets $\langle U_1, \ldots, U_n \rangle$ with U_i taken only from some open base for Y generate the same topology on $\mathcal{C}$. For a comprehensive overview on hyperspaces see [25].

Proof. ($\Rightarrow$) Let $\mathcal{C}$ denote the hyperspace of wL and $\mathcal{D}$ the hyperspace of X. As X is compact, $\mathcal{D}$ is compact as well. wL being in addition metric implies that $\mathcal{C}$ is compact metric (see for instance [12]). Take an arbitrary subcontinuum K of wL and consider a sequence $\langle K_i \rangle_{i \in \omega}$ of elements from L converging to K in $\mathcal{C}$. $\langle K_i \rangle_{i \in \omega}$ with K_i taken as subsets of X is then a sequence of subcontinua, so it has at least one cluster point, say C. Then C maps onto K so $w(e)$ is weakly confluent.

($\Leftarrow$) Let $e : L \to 2^X$ be the elementary embedding and $w(e)$ its Wallman representation.

We will show that for every subcontinuum A of wL and every open basic neighbourhood $A \in \langle U_1, \ldots, U_n \rangle$ such that for every i the complement H_i of U_i in wL is an element of L, there is a subcontinuum $E \in L \cap \langle U_1, \ldots, U_n \rangle$.

Pick an $F \in L$ with $A \subset \mathrm{int}(F)$ and $F \subset U_1 \cup \ldots \cup U_n$. Then $w(e)^{-1}[A] \subset F^X$, where F^X denotes the representation of F as a subset of X. If $w(e)$ is weakly confluent, then there is a component C of $w(e)^{-1}[A]$ such that C maps onto A. So $C \subset F^X \subset \bigcup_{i \in I} U_i^X$ and $C \cap U_i^X \neq \emptyset$ for $i = 1 \ldots, n$. Hence 2^X models

$$\exists\, G(\mathrm{conn}(G) \wedge G \sqcap \prod_{i \in I} H_i = \emptyset) \wedge \bigwedge_{j \in I} G \sqcap (\prod_{i \in I} H_i \sqcup H_j) \neq \emptyset. \tag{4.1}$$

In translation it means that there is a subcontinuum G of X that is an element of the open set $\langle U_1^X, \ldots, U_n^X \rangle$. By elementarity, there exists a subcontinuum E in L that fits into the open neighbourhood $\langle U_1, \dots, U_n \rangle$, which finishes the proof.

$\square$

The Question 1 remains open. However, it can now be generalized.

Question 7. *Is there a (non)-metric continuum that has span zero and is not chainable?*

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