Examples of nonholonomic systems[1]

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Abstract. The paper is concerned with non-holonomic mechanical systems, i.e. Lagrangian systems subjected to constraints on time, positions and velocities. Within a recent geometric theory, such systems are modelled on submanifolds of jet bundles over smooth fibred manifolds. Our aim is to study the geometric (reduced) equations of motion of some concrete constrained systems, interesting from the point of view of applications in physics. The geometric approach is appropriate and we show that can be effectively applied also to constraints non-linear in velocities. We investigate three concrete non-holonomic systems in detail: in each case we analyze the constraints, and find and solve the corresponding reduced equations on the constraint submanifold. For the first time we present an analysis and solution of the motion of a mechanical with nonlinear constraints.
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1 Introduction

Mechanical systems are usually studied by methods of the calculus of variations. However, mechanical systems with nonholonomic constraints are considered problematic, especially if non–integrable constraints and constraints non–linear in velocities are concerned [1], [9]. Although examples of this type are frequent in mechanics and engineering, solution is usually not available, and mechanical behavior of such systems is often surprising or even unpredictable. First attempts to understand nonholonomic systems go back to Chetaev, Neimark, Fufaev, Hertz, Kirchhoff and Hamel. In the last 15 years, in the context of rapid developments of differential geometry and global analysis and their applications to physics, this topic has been intensively studied and new mathematical methods and models to deal with nonholonomic systems have been invented. Among

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many authors studying nonholonomic systems with help of differential geometry let us mention Cantrijn, de Leon, Giachetta, Gracia, Krupková, Marsden, Massa, Pagani, Sarlet, and others (see e.g. references in [6]). Although now foundations of the theory of nonholonomic mechanical systems are quite well established, there is still a lack of *solved examples* even of systems appearing in mechanics and engineering (see e.g ([3], [4], [11]), that would justify the theory and contribute to enrich our knowledge of properties of such systems. In particular, up to now, no solution of a mechanical system with *nonlinear constraints* has been found and presented.

In this work we use the mathematical description of nonholonomic systems developed by O. Krupková [6], [8]. The theory is based on the model of nonholonomic constraints as submanifolds in jet bundles, and nonholonomic systems as exterior differential systems defined directly on "constraint submanifolds". The advantage of this approach is its applicability to general mechanical systems (not only variational) and all nonholonomic constraints, not only those linear in velocities, like models of other authors. Our aim is to study in detail selected examples of nonholonomic mechanical systems. In Section 2 we introduce notations and bring an overview of basic concepts and results on the geometry of unconstrained Lagrangian systems and Lagrangian systems subject to nonholonomic constraints in jet bundles. Section 3 is then devoted to the analysis of examples. The first two problems are mentioned but *not solved* in [2] and [10], respectively. Remarkably, the second one is an example of a constraint non-linear in velocities. Examples of non-linear constraints appear in the literature very exceptionally and only as a demonstration of the existence of such contraints, with no solution. The theory we use makes it possible to deal with such constraints. The third problem was suggested by Dr. Swaczyna (unpublished communication). In each of the three cases we analyze the constraints, and find the corresponding canonical distribution. Then we compute and solve the equations of motion (called *reduced equations*). In our examples this is a system of non-linear second order ordinary differential equations, it is solved numerically. Finally, graphs and comments are given.

2 Nonholonomic systems in fibred manifolds

In this section we recall basic structures and concepts used in the calculus of variations on fibered manifolds as introduced in [5], [6], [7] and [8]. Throughout the paper we consider a smooth fibered manifold $\pi: Y \to X$ with $\mathrm{dim} X = 1$, $\mathrm{dim} Y = m+1$, and its jet prolongations $\pi_r : J^r Y \to X$, $r = 1, 2$.

A differential form η is called *closed* if $\mathrm{d}\eta = 0$. We shall use the ideal of contact forms on $J^1 Y$, locally generated by 1-forms

$$\omega^\sigma = \mathrm{d}q^\sigma - \dot{q}^\sigma \mathrm{d}t, \quad 1 \leq \sigma \leq m.$$

A q-form η is called *horizontal* if its contraction $i_\xi \eta$ by every vertical vector field

ξ vanishes. A contact q-form η is called 1-*contact* if for every vertical vector field ξ the form $i_\xi \eta$ is horizontal, and k-*contact* if $i_\xi \eta$ is $(k-1)$-contact, $k = 2, \ldots, q$. If $f\,(t, q^\sigma, \dot{q}^\sigma)$ is a function we write

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma}\dot{q}^\sigma + \frac{\partial f}{\partial \dot{q}^\sigma}\ddot{q}^\sigma, \quad \frac{\bar{\mathrm{d}}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma}\dot{q}^\sigma.$$

Here and in what follows, summation over repeated indices is always assumed.

By a *distribution* on J^1Y we mean a mapping Δ assigning to every point $z \in J^1Y$ a vector subspace Δ_z of the vectore space T_zJ^1Y, i.e. $\Delta : J^1Y \ni z \mapsto \Delta_z \subset T_zJ^1Y$. A distribution is said to be of a *constant rank* if $\dim \Delta_z$ does not depend on z. By the *annihilator* of Δ we shall mean the module of all 1-forms η_k on open subsets of J^1Y such that $i_{\xi_k}\eta_k = 0$ for every vector field ξ_i belonging to Δ. A section $\delta : X \to J^1Y$ is called *integral section* of Δ if for every 1-form η belonging to the annihilator of Δ, $\delta^*\eta = 0$.

A first order *Lagrangian* is defined to be a horizontal 1-form λ on J^1Y, a *dynamical form* of order 2 is a 1-contact 2-form, horizontal with respect to the projection onto Y. In fibered coordinates, $\lambda = L\,\mathrm{d}t$, $E = E_\sigma \omega^\sigma \wedge \mathrm{d}t$. A dynamical form is called *locally variational* if around each point in J^1Y there is a Lagrangian λ such that E is the Euler–Lagrange form of λ.

By a *first-order mechanical system* we shall mean the equivalence class of Lepagean 2-forms on J^1Y, associated to a dynamical form E. A mechanical system is denoted by $[\alpha]$ and the class of the corresponding dynamical distributions by $[\Delta_\alpha]$. A mechanical system $[\alpha]$ is called *regular* if there exists a dynamical distribution $\Delta_\alpha \in [\Delta_\alpha]$ such that $\operatorname{rank} \Delta_\alpha = 1$. A mechanical system related with a locally variational form is called a *Lagrangian system. Euler–Lagrange equations* then have the form

$$J^1\gamma^* i_\xi \alpha = 0, \tag{1}$$

where ξ is a π_1-vertical vector field on J^1Y, and α is any representative of the class $[\alpha]$, and the regularity condition has the form

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}\right) \neq 0,$$

where L is a first-order Lagrange function corresponding to E.

By a *nonholonomic constraint submanifold* in J^1Y we mean a fibered submanifold $\pi_{1,0} : Q \to Y$ of the fibered manifold $\pi_{1,0} : J^1Y \to Y$. We denote by ι the canonical embedding of Q into J^1Y and suppose $\operatorname{codim} Q = k < m$. Then around every point $x \in Q$ there is a coordinate system (U, χ), $\chi = \left(t, q^1, \ldots, q^{m-k}, \dot{q}^1, \ldots, \dot{q}^{m-k}, f^1, \ldots, f^k\right)$ on J^1Y such that the restriction of Q to U can be given by equations

$$f^i = 0, \quad 1 \leq i \leq k, \tag{2}$$

where

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = k. \tag{3}$$

Equivalently, the submanifold Q can be locally expressed by equations in normal form as follows:

$$\dot{q}^{m-k+i} = g^i\left(t, q^\sigma, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^{m-k}\right), \quad 1 \leq i \leq k. \tag{4}$$

Let $Q \subset J^1Y$ be a constraint submanifold. A section γ of π defined on an open set $I \subset X$ is called a *holonomic path in* Q if $J^1\gamma(x) \in Q$ for every $x \in I$.

A nonholonomic constraint $Q \subset J^1Y$ is called *simple* if it can be represented by a distribution D on the manifold Y. Simple nonholonomic constraints are just constraints affine in the velocities. Special cases of simple nonholonomic constraints are *semiholonomic constraints*, representable by completely integrable distributions in Y, and so-called *Chaplygin constraints*, given by equations

$$\dot{q}^{m-k+i} = \sum_{s=1}^{m-k} b^i_s\left(t, q^1, \ldots, q^{m-k}\right) \dot{q}_s, \tag{5}$$

$1 \leq i \leq m$. If the constraints are Chaplygin constraints and the Lagrangian does not depend on q^{m-k+i}, $1 \leq i \leq m$, the constrained system is called *Chaplygin system*.

Every constraint manifold Q naturally caries a distribution, called the *canonical distribution*, denoted by C. It is annihilated by the following system of k linearly independent local 1-forms

$$\iota^*\varphi^i = -\frac{\partial g^i}{\partial \dot{q}^l}\,\omega^l + \mathrm{d}q^{m-k+i} - g^i\,\mathrm{d}t, \quad 1 \leq i \leq k, \tag{6}$$

called *canonical constraint* 1*-forms*. From the point of view of physics, the canonical distribution represents possible generalized displacements (in the space of positions and velocities). The ideal in the exterior algebra on Q generated by canonical constraint 1-forms (6) is called the *constraint ideal* and denoted by $I\left(C^0\right)$. The forms belonging to the constraint ideal are called *constraint forms*.

Let E be a dynamical form and $[\alpha]$ the corresponding mechanical system. With help of the nonholonomic constraint structure (Q, C) one can construct a new mechanical system directly on the constraint submanifold Q of J^1Y. It is the equivalence class $[\alpha_Q]$ on Q, where $\alpha_Q = \iota^*\alpha + F + \varphi_{(2)}$, where F is a 2-contact 2-form and $\varphi_{(2)}$ is a constraint 2-form. Equations of motion of the constrained system $[\alpha_Q]$ then have the following intrinsic form

$$J^1\gamma^* i_\xi \alpha_Q = 0 \tag{7}$$

for every vector field $\xi \in C$, where α_Q is any 2-form belonging to the class $[\alpha_Q]$; they are called *reduced equations*.

We note that the 2-form α_Q need not be closed and moreover, in general, there is no closed 2-form in the class $[\alpha_Q]$. A constrained system $[\alpha_Q]$ is Lagrangian if and only if there is a closed 2-form belonging to the class $[\alpha_Q]$. If $\lambda = L\,dt$ is a Lagrangian for E and

$$\theta_\lambda = L\,\mathrm{d}t + \frac{\partial L}{\partial \dot{q}^\sigma}\omega^\sigma$$

is its Cartan form then the corresponding unconstrained Lagrangian system is $[d\theta_\lambda]$, and the constrained system is the equivalence class $[\iota^* d\theta_\lambda]$. Reduced equations of motion then take the coordinate form as follows:

$$f^i \circ J^1\gamma = 0, \qquad \left(A'_l + \sum_{s=1}^{m-k} B'_{l,s}\, \ddot{q}^s \right) \circ J^2\gamma = 0, \tag{8}$$

$1 \le i \le k$, $1 \le l \le m-k$, where

$$A'_l = \frac{\partial \bar{L}}{\partial q^l} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial \bar{L}}{\partial \dot{q}^l}\right) + \tag{9}$$

$$+ \left(\frac{\partial L}{\partial \dot{q}^{m-k+j}} \circ \iota\right) \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial g^j}{\partial \dot{q}^l}\right) - \frac{\partial g^j}{\partial q^l} - \frac{\partial g^j}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} \right],$$

$$B'_{l,s} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} + \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \circ \iota\right) \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s}, \tag{10}$$

and $\bar{L} = L \circ \iota$, i.e.

$$\bar{L}\left(t, q^\sigma, \dot{q}^l\right) = L\left(t, q^\sigma, \dot{q}^l, g^i\left(t, q^\sigma, \dot{q}^l\right)\right). \tag{11}$$

These equations are regular if

$$\det\left(\frac{\partial \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} - \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}} \circ \iota\right) \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s} \right) \neq 0. \tag{12}$$

A constrained system of regular mechanical system need not be regular.

3 Examples

In this section we discuss three examples of nonholonomic constrained Lagrangian systems. We describe the constraints, find reduced equations of motion, and solve them numerically in the computer program *wxMaxima* by a method of Runge-Kutta type of fourth order.

The first example is chosen from the article [2], p. 27, where it is mentioned as an example of a Chaplygin system, without solution.

Example 1. *Let us find trajectories of a "free particle" moving in* $\mathbb{R}^3$*, and subject to the nonholonomic constraint*

$$f \equiv \dot{z} - yx\dot{x} = 0. \tag{13}$$

The unconstrained mechanical system has three degrees of freedom. Let the mass of the particle be equal to one. We denote by (t) the coordinate on $X = \mathbb{R}$, by (t, x, y, z) the fiber coordinates on $Y = \mathbb{R} \times \mathbb{R}^3$ and by $(t, x, y, \dot{x}, \dot{y}, \dot{z})$ the associated coordinates on $J^1Y = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$. The Lagrange function is the partice's kinetic energy

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right).$$

The constraint everywhere satisfies the rank condition (3):

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = \begin{pmatrix} -xy & 0 & 1 \end{pmatrix} = 1.$$

Hence the constraint submanifold $Q \subset \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ is 6-dimensional and is given by equation in normal form

$$g = xy\dot{x}.$$

On the constraint submanifold Q the Lagrange function $\bar{L}$ is

$$\bar{L} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + (xy\dot{x})^2\right).$$

Computing the coefficients A'_l by (9), respectively $B'_{l,s}$ by (10) we obtain

$$\begin{aligned} A'_1 &= -xy^2\dot{x}^2 - x^2y\dot{x}\dot{y}, & B'_{1,1} &= -1 - x^2y^2, \\ A'_2 &= 0, & B'_{2,2} &= -1, \\ & & B'_{1,2} &= B'_{2,1} = 0. \end{aligned}$$

The regularity condition (12)

$$\det \begin{pmatrix} -\left(1 + x^2y^2\right) & 0 \\ 0 & -1 \end{pmatrix} = 1 + x^2y^2 \neq 0$$

is satisfied at each point in Q.

The reduced equations of motion of the constrained system take the form:

$$\begin{aligned} -xy\left(y\dot{x} + x\dot{y}\right) - \left(1 + x^2y^2\right)\ddot{x} &= 0, \\ -\ddot{y} &= 0. \end{aligned}$$

For numerical calculation it is necessary to convert the system of equations to normal form:

$$\begin{aligned}\ddot{x} = & -\tfrac{xy^2\dot{x}^2}{1+x^2y^2} - \tfrac{x^2y\dot{x}\dot{y}}{1+x^2y^2},\\ \ddot{y} = & \qquad 0.\end{aligned}$$

Let us choose initial conditions as follows: Let the particle move from the beginning of the system of coordinates $[0, 0, 0]$ with initial velocity $\dot{x}(0) = 1$ and $\dot{y}(0) = 1$. For numerical calculation use the step $h = 0.01$. The resulting motion of the particle is illustrated by Figure 1.

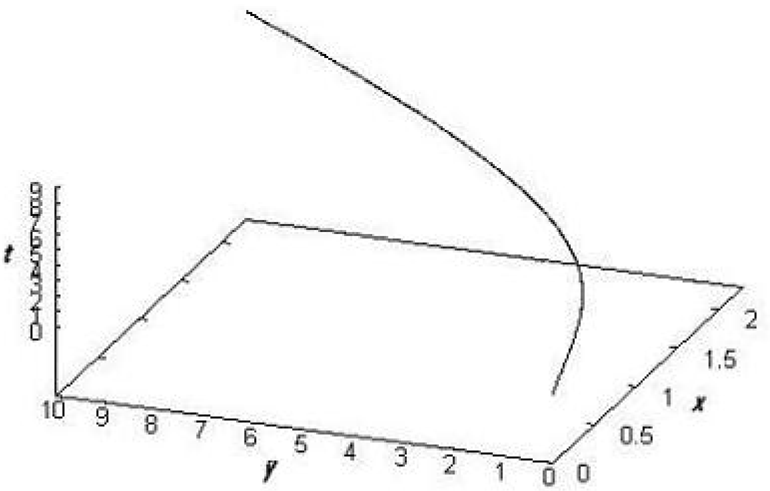

Fig. 1.

The following figures show the projection of the trajectory in the xy and xz plane.

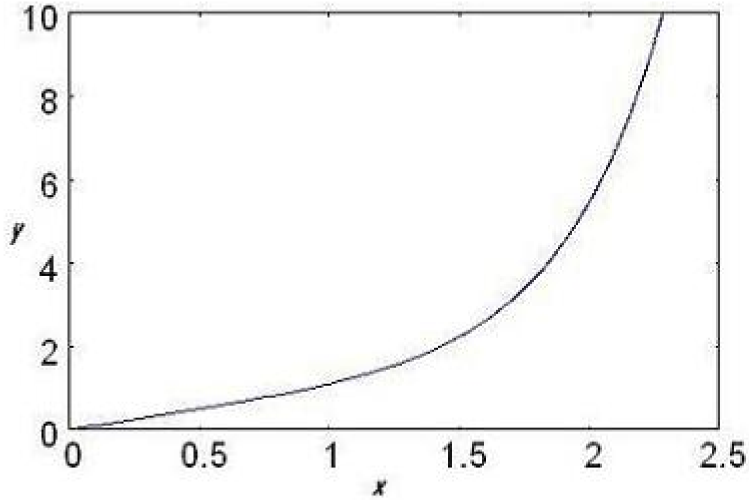

Fig. 2. *Projection of the trajectory in the xy plane.*

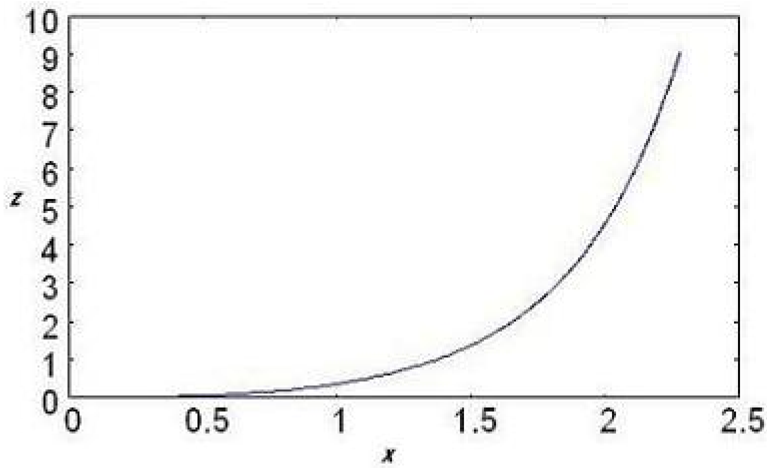

Fig. 3. *Projection of the trajectory in the xz plane.*

Figures 2 and 3 are very similar, however we can see that the value of y is growing faster than the value of z. For example, if $x = 1.5$ then $y = 2.25$, while $z = 1.4$.

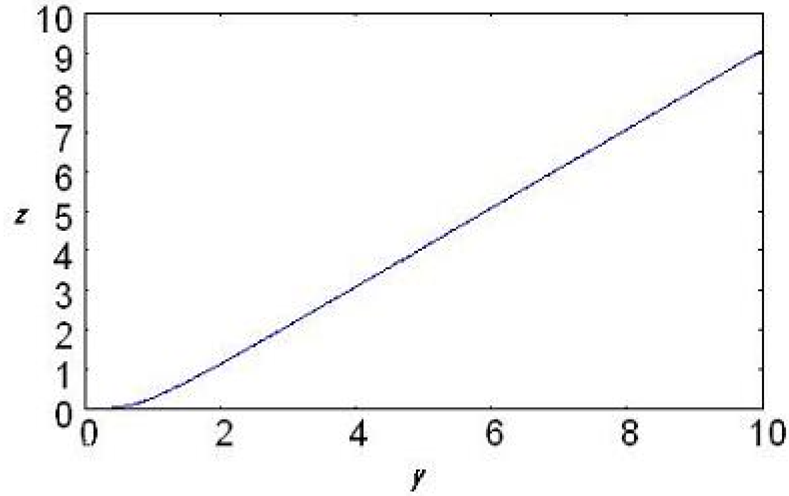

Fig. 4. *Projection of the trajectory in the yz plane.*

The last picture just confirms the fact, most on the interval $(0, 2)$.

We can conclude that, roughly speaking, the particle moves away from the axis x by twisting around z and "almost" as fast rising up.

The next example is an example of a mechanical system with a nonlinear constraint on velocities. It is chosen from the article [10], p. 223, where, however, it is only mentioned as a model example, with a comment that a solution is not available.

Example 2. *Let us solve the motion equations of a particle of mass m moving in $\mathbb{R}^3$ in the field of the gravitational potential mGz, and subject to the nonholonomic nonlinear constraint*

$$f \equiv \dot{x}^2 + \dot{y}^2 - a^2\dot{z}^2 = 0, \tag{14}$$

where $a \neq 0$ is a constant.

As above, we denote by (t) the coordinate on $X = \mathbb{R}$, by (t, x, y, z) the canonical coordinates on $Y = \mathbb{R} \times \mathbb{R}^3$ and by $(t, x, y, z, \dot{x}, \dot{y}, \dot{z})$ the associated coordinates on $J^1Y = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$. The Lagrange function has the form

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mGz,$$

where m is the mass of the particle and $G \doteq 10$ ms^{-2} is the gravitational acceleration.

Condition (3) rank $\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = \begin{pmatrix} 2\dot{x} & -2\dot{y} & -2a^2\dot{z} \end{pmatrix} = 1$ is satisfied at the points where $\vec{v} = (\dot{x}, \dot{y}, \dot{z}) \neq 0$. The constraint (14) is expressed in normal form

(4) by equations

$$g = \pm \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{a}.$$

These equations define a 6-dimensional disconnected submanifold Q of $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$.

We shall consider the connected component Q_+ of Q where $g = \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{a}$. The manifold Q_+ is a submanifold of the open subset of $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ where $\dot{z} > 0$. The Lagrange function $\bar{L}$ is

$$\bar{L} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \left(\frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{a} \right)^2 \right) - mGz.$$

Computing the coefficients A'_l by (9), respectively $B'_{l,s}$ by (10) we obtain

$$A'_1 = -mG \frac{\dot{x}}{a\sqrt{\dot{x}^2+\dot{y}^2}}, \quad B'_{1,1} = m \left(-1 - \frac{1}{a^2} + \frac{\dot{y}^2}{a^2(\dot{x}^2+\dot{y}^2)} \right),$$

$$A'_2 = -mG \frac{\dot{y}}{a\sqrt{\dot{x}^2+\dot{y}^2}}, \quad B'_{2,2} = m \left(-1 - \frac{1}{a^2} + \frac{\dot{x}^2}{a^2(\dot{x}^2+\dot{y}^2)} \right),$$

$$B'_{1,2} = B'_{2,1} = -m \frac{\dot{x}\dot{y}}{a^2(\dot{x}^2+\dot{y}^2)}.$$

The regularity condition (12) is reduced to

$$\det \begin{pmatrix} \left(-1 - \frac{1}{a^2} + \frac{\dot{y}^2}{a^2(\dot{x}^2+\dot{y}^2)} \right) & -\frac{\dot{x}\dot{y}}{a^2(\dot{x}^2+\dot{y}^2)} \\ \left(-1 - \frac{1}{a^2} + \frac{\dot{x}^2}{a^2(\dot{x}^2+\dot{y}^2)} \right) & -\frac{\dot{x}\dot{y}}{a^2(\dot{x}^2+\dot{y}^2)} \end{pmatrix} =$$

$$1 + \frac{2}{a^2} + \frac{1}{a^4} + \frac{\dot{x}^2 - \dot{y}^2}{a^2 \left(\dot{x}^2 + \dot{y}^2 \right)} - \frac{\dot{x}^2 + \dot{y}^2}{a^4 \left(\dot{x}^2 + \dot{y}^2 \right)} \neq 0,$$

and is satisfied. The reduced equations of motion of the constrained system are regular and have the form:

$$-\frac{mG\dot{x}}{a\sqrt{\dot{x}^2 + \dot{y}^2}} + m \left(-1 - \frac{1}{a^2} + \frac{\dot{y}^2}{a^2 \left(\dot{x}^2 + \dot{y}^2 \right)} \right) \ddot{x} - \frac{m\dot{x}\dot{y}}{a^2 \left(\dot{x}^2 + \dot{y}^2 \right)} \ddot{y} = 0,$$

$$-\frac{mG\dot{y}}{a\sqrt{\dot{x}^2 + \dot{y}^2}} + m \left(-1 - \frac{1}{a^2} + \frac{\dot{x}^2}{a^2 \left(\dot{x}^2 + \dot{y}^2 \right)} \right) \ddot{y} - \frac{m\dot{x}\dot{y}}{a^2 \left(\dot{x}^2 + \dot{y}^2 \right)} \ddot{x} = 0.$$

For numerical calculation it is necessary to convert the system of equations to normal form:

$$\ddot{x} = \frac{a^3 G}{(a^2+1)(2-a^2)} \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}},$$

$$\ddot{y} = -\frac{aG\left(1+a^2\right)}{(2-a^2)} \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{y}}.$$

We shall again find a solution numerically using the program wxMaxima and Runge-Kutta methods. We choose $a = 2$, and select the initial conditions as follows: the particle moves from the beginning of the coordinate system $[0, 0, 0]$ with initial velocity $\dot{x}(0) = 2$ and $\dot{y}(0) = 2$. The step of the method will be $h = 0.01$. The resulting trajectory of the particle is illustrated by the following figures.

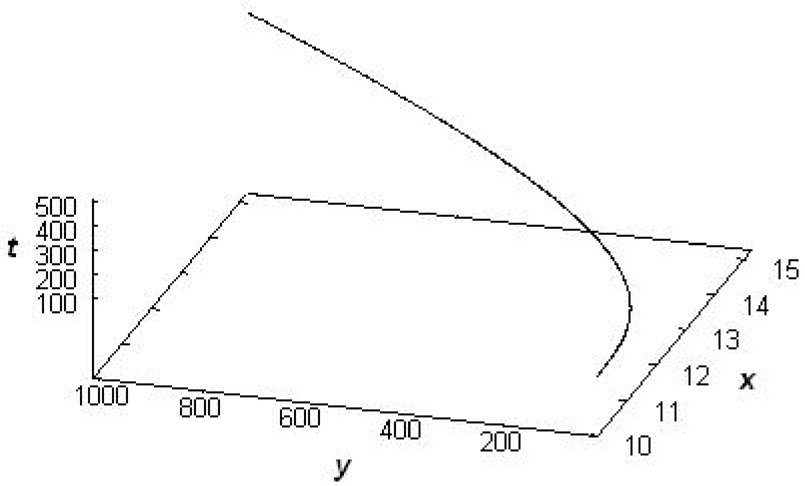

Fig. 5.

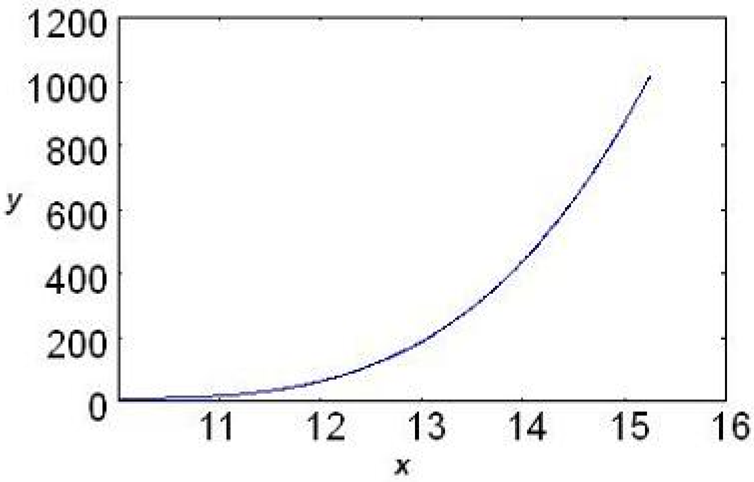

Fig. 6. *Projection of the trajectory in the xy plane.*

We can see that y grows significantly faster than x. For example, for $x = 12$ we get $y = 66$, for $x = 15$ we get $y = 858$.

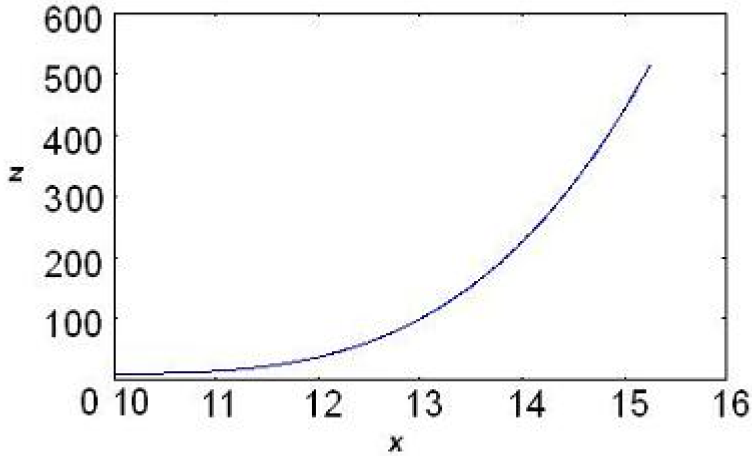

Fig. 7. *Projection of the trajectory in the xz plane.*

Values of z grow faster than that of x, but not so much as in the previous case. $x = 12$ corresponds to $z = 39$, however for $x = 15$ we have $y = 446$. The last figure shows the projection to the yz plane.

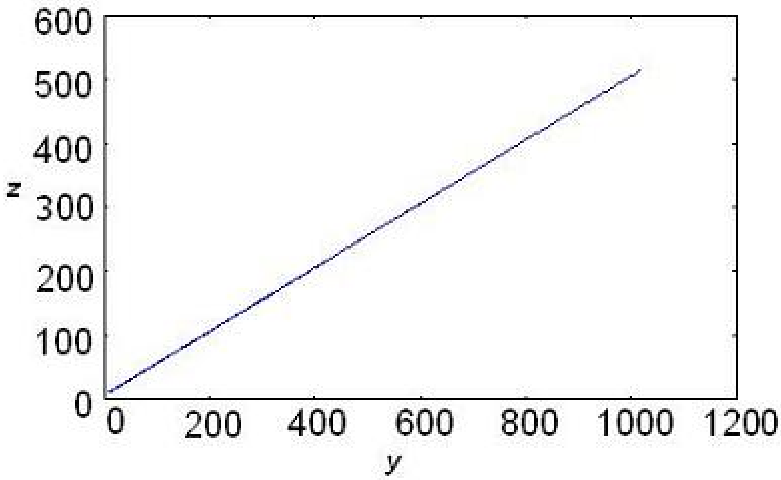

Fig. 8. *Projection of the trajectory in the yz plane.*

We can see that the dependence of z on y is almost linear. For examle, for

$y = 40$ we have $z = 26$ and for $y = 200$ we have $z = 106$. For higher values, e.g. $y = 800$ we get $z = 406$. We can conclude that y grows approximately twice faster than z. The overall movement of the particle goes away from the x-axis and rises up twice slower.

The following example was suggested by M. Swaczyna.

Example 3. *Consider a particle of mass m in the vertical plane xy in the field of the potential mGy which is thrown from the origin of the coordinate system with the initial velocity $\vec{v}_0$. We neglect the air resistance. From a location on the x-axis another particle is sent. Find the trajectory of the second particle, if its velocity vector at any time is directed to the first particle.*

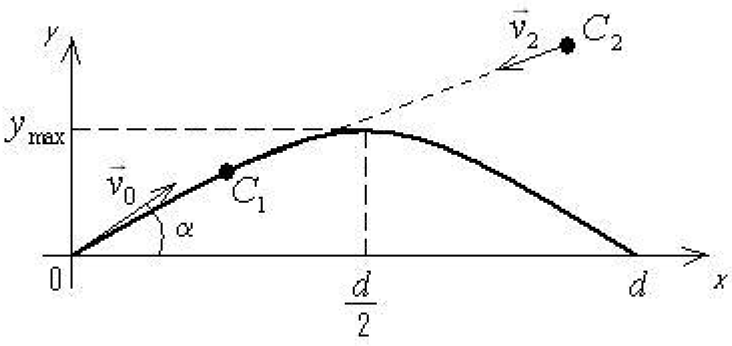

Fig. 9.

The trajectory of the first particles is known, it is a standard throw. We denote by $\vec{v}_{(0)}$ the vector of the initial velocity of the first particle C_1, $\vec{v}_2$ the vector of the initial velocity of the second particle C_2, and α the angle between the x-axis and $\vec{v}_{(0)}$. Next denote d the length the throw and y_{max} its maximum height (Fig. 9). For the first particle trajectory we have the parametric expressions

$$\begin{aligned} \xi(t) &= v_0 t \cos\alpha, \\ \eta(t) &= v_0 t \sin\alpha - \tfrac{1}{2} G t^2, \end{aligned} \tag{15}$$

where $G \doteq 10$ ms^{-2} is the gravitational acceleration. From the condition that the velocity vector of the second particle is directed at any time to the first particle we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k\,(y\,(t) - \eta\,(t))}{k\,(x\,(t) - \xi\,(t))}. \tag{16}$$

Substituting ξ and η from (15) to (16) gives us

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\left(t\right) - v_0 t \sin\alpha + \frac{1}{2}Gt^2\left(t\right)}{x\left(t\right) - v_0 t \cos\alpha\left(t\right)}.$$

Denoting $k_1 = v_0 \cos\alpha$, $k_2 = v_0 \sin\alpha$, we get the nonholonomic constraint

$$f \equiv \dot{y}\left(x - k_1 t\right) - \dot{x}\left(y - k_2 t + \frac{1}{2}Gt^2\right) = 0. \tag{17}$$

The Lagrange function in this example has the form

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mGy$$

and is defined on $J^1Y = \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$.

The rank condition (3) for the constraint takes the form

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = \begin{pmatrix} y - k_2 t + \frac{1}{2}Gt^2 & x - k_1 t \end{pmatrix} = 1,$$

and is satisfied at the points $x \neq k_2 t - \frac{1}{2}Gt^2$, $y \neq k_1 t$. The constraint (17) is expressed in normal form by (4)

$$g = \frac{\dot{x}\left(x - k_1 t\right)}{\left(y - k_2 t + \frac{1}{2}Gt^2\right)}.$$

For the Lagrange function $\bar{L}$ we get by (11)

$$\bar{L} = \frac{1}{2}m\left(\dot{x}^2 + \left(\frac{\dot{x}\left(x - k_1 t\right)}{\left(y - k_2 t + \frac{1}{2}Gt^2\right)}\right)^2\right) - mGy.$$

Computing the coefficients A'_l by (9), respectively $B'_{l,s}$ by (10):

$$A'_1 = -mG\frac{y - t\left(k_1 - \frac{1}{2}Gt\right)}{x - k_2 t} - m\frac{\dot{x}\left(y - t\left(k_1 - \frac{1}{2}Gt\right)\right)\left(Gt - k_1\right)}{\left(x - k_2 t\right)^2} -$$

$$- mk_2\frac{\left(y - t\left(k_1 - \frac{1}{2}Gt\right)\right)^2}{\left(x - k_2 t\right)^3} - m\frac{\dot{x}\dot{y}\left(y - t\left(k_1 - \frac{1}{2}Gt\right)\right)}{\left(x - k_2 t\right)^2} +$$

$$+ m\frac{\dot{x}^2\left(y - t\left(k_1 - \frac{1}{2}Gt\right)\right)^2}{\left(x - k_2 t\right)^3},$$

$$B'_{1,1} = -m\left(1 + \frac{\left(y - t\left(k_1 - \frac{1}{2}Gt\right)\right)^2}{\left(x - k_2 t\right)^2}\right).$$

The regularity condition (12) now reads

$$-m\left(1+\frac{\left(y-t\left(k_1-\frac{1}{2}Gt\right)\right)^2}{\left(x-k_2t\right)^2}\right)\neq 0.$$

The reduced equation of motion of the constrained system is now only one, is regular and has the form:

$$\begin{aligned}m\dot{x}^2\left(x-k_1t\right)-\tfrac{2(x-k_1t)}{\left(y-t\left(k_2-\frac{1}{2}Gt\right)\right)^3}-mG\left(x-k_1t\right)+\\ +\tfrac{m\dot{x}(x-k_1t)\left(-k_1\left(y-t\left(k_2-\frac{1}{2}Gt\right)\right)^2+x-k_1t\right)}{\left(y-t\left(k_2-\frac{1}{2}Gt\right)\right)^3}-\\ -2m\dot{x}\left(x-k_1t\right)+2mk_1\dot{x}\left(x-k_1t\right)-\left(1+\left(x-k_1t\right)\right)\ddot{x}=0.\end{aligned}\tag{18}$$

We convert this equation to normal form

$$\ddot{x}=mk_2\frac{\dot{x}\left(y-t\left(k_1-\frac{1}{2}Gt\right)\right)^2}{\left(y-t\left(k_1-\frac{1}{2}Gt\right)\right)^2+\left(x-k_2t\right)^2}-m\frac{\dot{x}\left(y-t\left(k_1-\frac{1}{2}Gt\right)\right)\left(Gt-k_1\right)}{\left(y-t\left(k_1-\frac{1}{2}Gt\right)\right)^2+\left(x-k_2t\right)^2}-$$
$$-mG\frac{\left(y-t\left(k_1-\frac{1}{2}Gt\right)\right)\left(x-k_2t\right)}{\left(y-t\left(k_1-\frac{1}{2}Gt\right)\right)^2+\left(x-k_2t\right)^2},$$

and solve the system of two equations - the above one together with the equation of the constraint. Again we proceed numerically using the program wxMaxima and Runge-Kutta methods. Let the first particle move from the origin of the coordinate system $[0,0]$ with the speed $|\vec{v}_0|=20$, at the angle $\alpha=45°$. Let the second particle move from the point $[40,20]$ with the initial speed $|\vec{v}_2|=1$. Let the mass $m=1$ and choose the step of the method $h=0.01$. The resulting motion of the second particle is shown on Figure 10.

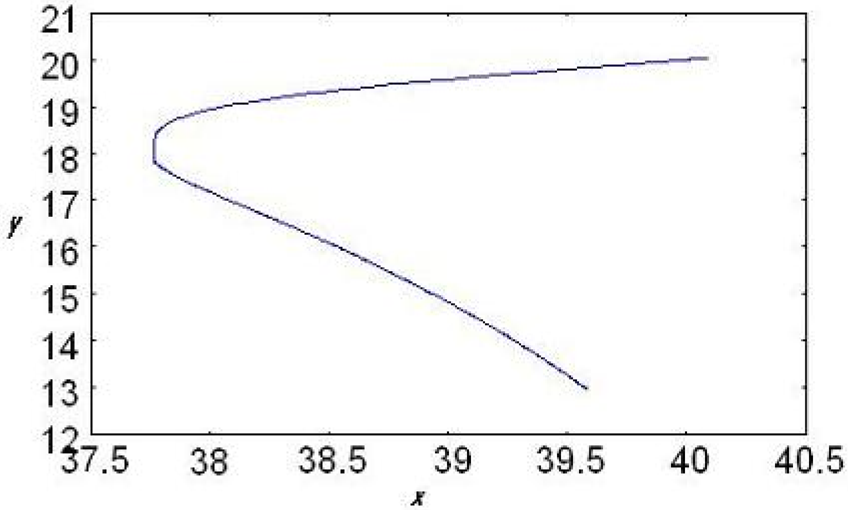

Fig. 10. *Trajectory of the second particle.*

Initial conditions for the second particles C_2 must be carefully chosen. Its motion is subjected to not only to the nonholonomic condition, but also to gravity, which forces the particle rapidly decline. Thus, for example, letting the particle C_2 to strat from the point $[5, 20]$ with a fast speed, gives no reasonable solution, because the particle falls down.

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