Embeddings in the Spaces of Hölder Functions

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1 Introduction

The purpose of this text is to show some properties, especially those involving continuous and compact embeddings, of the spaces of Hölder continuous functions on a one-dimensional compact interval. These fuctions, which we will call simply Hölder functions for short, are usually defined in another way than we are going to do. This definition contains a norm involving the α^{th} power function for some $\alpha \in (0, 1)$ (this approach is metioned at the very end of the text). However, that definition can be easily generalized in order to obtain much wider class of functions and this is the approach we will use.

In our case, the space of Hölder functions is established using a "Hölder norm" employing some real-valued function which we will denote by φ. Hence, properties of the space depend on the properties of the function φ. Our main goal is to describe when a Hölder space is continuously or even compactly embedded in another one. The answer to this question can be obtained by comparing the functions which "generate" the respective norms of those spaces. It will be shown, for instance, that the spaces can be in fact the same although the norm-generating functions are different. Generally, we will provide equivalent characterizations of the continuous and compact embeddings of the Hölder spaces.

2 Preliminaries

First of all, let us summarize the basic notation and statements which will be used later in the text.

Throughout the text, K denotes the compact unit interval $\langle 0, 1\rangle \subset \mathbb{R}$.

Definition 1. Let (X, ϱ) be a metric space and A a subset of X. $\bar{A}$ then denotes the *closure* of A which is the smallest closed subset of X containing A.

Proposition 2. *Let (X, ϱ) be a metric space and A a subset of X. Then*

$$\bar{A} = \{x \in X \colon \exists \{x_n\} \subset X, \varrho\,(x_n, x) \xrightarrow{n\to\infty} 0\}.$$

Definition 3. Let (X, ϱ) be a metric space. A subset $A \subset X$ is called *compact* if every sequence $\{x_n\}_{n=1}^{\infty} \subset A$ has a converging subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ whose limit x lies in A. (Symbolically: $\varrho\,(x_{n_k}, x) \xrightarrow{k\to\infty} 0$.)
A is called *relatively compact* if $\bar{A}$ is compact.

Theorem 4. *A subset A of a metric space (X, ϱ) is compact if and only if for each open cover of A there exists a finite subcover.*

The condition of all open covers having finite subcovers is a general definition of a set's compactness in topological spaces. The property introduced by our Definition 3 is then called *sequential compactness.* The theorem above says that in the case of metric spaces the two definitions are equivalent.

Definition 5. Let $(X, \|\cdot\|_X)$ is a normed space ($\|\cdot\|_X$ denotes its norm). A subset $A \subset X$ is called *bounded* if there exists a real constant C such that

$$\|x\|_X < C \quad \forall\, x \in A.$$

In other words,

$$\sup_{x\in A} \|x\|_X < \infty.$$

In the context of function spaces, it is quite common to call the bounded subsets of a (normed) function space to be *equibounded* to distinguish between boundedness of a single function and boundedness of a set of functions as elements of the function space. Anyway, we will use the term "bounded" in both cases since it will be clear if it describes a property of a single function or a set of functions. However, the norm in which is a set bounded can be emphasized to prevent confusion when dealing with more function spaces and their respective norms.

Definition 6. Let (X, ϱ) be a normed space. The symbol $C(X)$ ($C^0(X)$ alternatively) denotes the space of continuous functions $f\colon X \to \mathbb{R}$ equipped with the *uniform norm* defined as

$$\|f\| := \sup_{x \in X} |f(x)|.$$

Next, we define

$$C^n(X) := \{f\colon X \to \mathbb{R}, f^{(n)} \text{ exists and } f^{(n)} \in C(X)\}.$$

The norm of $C^n(X)$ is defined as

$$\|f\|_n := \|f\| + \sum_{k=1}^{n} \|f^{(n)}\|$$

for a function $f\colon X \to \mathbb{R}$.

This text will deal in particular with $C(K)$ and $C^n(K)$. In this case the following proposition is true:

Proposition 7. *Let X be a compact metric space. Then $C(X)$ is a Banach space.*

The continuous and compact embeddings, the topic of this text, are defined as follows:

Definition 8. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. We say that the space X is *continuously embedded into* Y and denote this by

$$X \hookrightarrow Y$$

if it holds that $X \subset Y$ (as sets) and there exists $b > 0$ such that

$$\|x\|_Y \leq b\|x\|_X \quad \forall\, x \in X.$$

Moreover, we say that the spaces X and Y are *equivalent*, written

$$X \cong Y$$

if $X = Y$ (as sets) and there exist $a, b > 0$ such that

$$a\|x\|_X \leq \|x\|_Y \leq b\|x\|_X \quad \forall\, x \in X.$$

Finally, we say that the space X is *compactly embedded into* Y and denote this by

$$X \hookrightarrow\hookrightarrow Y$$

if $X \subset Y$ (as sets) and every bounded set in $(X, \|\cdot\|_X)$ is relatively compact in $(Y, \|\cdot\|_Y)$.

In the study of compact embeddings of some spaces, a description of their (relatively) compact subsets is always useful. In the case of continuous functions over a compact metric space, such a criterion is provided by the Arzelà-Ascoli theorem.

Definition 9. Let (X, ϱ) be a metric space and $x_0 \in X$. A subset $M \subset C(X)$ is called *equicontinuous at* x_0, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon$$

for all $f \in M$ and all $x \in X$ such that $\varrho(x, x_0) < \delta$.
M is called *equicontinuous* if it is equicontinuous at every $x_0 \in X$.

Theorem 10 (Arzelà-Ascoli)**.** *Let (X, ϱ) be a compact metric space. A subset of $C(X)$ is relatively compact in $C(X)$ if and only if it is bounded and equicontinuous.*

The norm of Hölder functions is in fact quite similar to the norm of the Lipschitz ones. Not only for this reason the definition below is presented:

Definition 11. Let $F\colon K \to \mathbb{R}$ be a real-valued function. The *Lipschitz seminorm* of F is defined by

$$|F|_L := \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|F(x) - F(y)|}{|x - y|}.$$

If $|F|_L < \infty$, then F is called *Lipschitz function* on K. We denote by $\mathrm{Lip}(K)$ the space of all Lipschitz functions with the (Lipschitz) norm

$$\| \cdot \|_L := \| \cdot \| + | \cdot |_L.$$

All the theorems and propositions above are well-known, however, their proofs can be found, for instance, in [1] if needed.

Having introduced the preliminaries, we can proceed to the main part of the work.

3 Continuous embeddings of the Hölder spaces

It was said that the Hölder functions would be introduced in a generalized sense so we start with proper definitions.

Definition 12. We denote by H the set of all functions φ which satisfy the following conditions:

(H1) $\varphi\colon K \to \langle 0, \infty)$ is non-decreasing on K,

(H2) $\lim_{t\to 0+} \varphi(t) = 0$,

(H3) $\lim_{t\to 0+} \frac{\varphi(t)}{t} = \infty$.

Definition 13. Assume that $\varphi \in H$. Then, for a function $f\colon K \to \mathbb{R}$ we define the *φ-Hölder seminorm of f* as

$$|f|_\varphi := \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x)-f(y)|}{\varphi(|x-y|)}.$$

We denote by $C^{0,\varphi}(K)$ the space of all real-valued functions f on K such that $|f|_\varphi < \infty$ equipped with the norm

$$\|\cdot\|_\varphi := \|\cdot\| + |\cdot|_\varphi. \tag{1}$$

A function $f \in C^{0,\varphi}(K)$ is called a *φ-Hölder function on K*.

We will study the question when one Hölder space is continuously embedded in another one, depending on properties of the functions from H which generate the respective Hölder seminorms of those spaces.

Theorem 14. *Consider functions $\varphi, \psi \in H$ and denote*

$$C_1 := \liminf_{t\to 0+} \frac{\psi(t)}{\varphi(t)}, \quad C_2 := \limsup_{t\to 0+} \frac{\psi(t)}{\varphi(t)}.$$

Then the following statements hold:

(i) *If $C_1 > 0$, then $C^{0,\varphi}(K) \hookrightarrow C^{0,\psi}(K)$.*

(ii) *If $C_2 < \infty$, then $C^{0,\psi}(K) \hookrightarrow C^{0,\varphi}(K)$.*

(iii) *If $0 < C_1 \le C_2 < \infty$, then $C^{0,\varphi}(K) \cong C^{0,\psi}(K)$.*

Proof. (i) Suppose that

$$\liminf_{t\to 0+} \frac{\psi(t)}{\varphi(t)} = C_1 > 0. \tag{2}$$

From (2), there exists $\delta > 0$ such that

$$C_1 \leq \frac{\psi(t)}{\varphi(t)} \quad \forall\, t \in (0,\delta). \tag{3}$$

Consider a function $f\colon K \to \mathbb{R}$. We know that both φ and ψ are non-negative and non-decreasing functions, hence we have for all $t \in \langle \delta, 1\rangle$:

$$\frac{\psi(\delta)}{\varphi(1)} \leq \frac{\psi(t)}{\varphi(t)}.$$

Thus, for all $x, y \in K$ such that $|x-y| \geq \delta$ and $f(x) \neq f(y)$:

$$\frac{\psi(\delta)}{\varphi(1)} \leq \frac{|f(x)-f(y)|}{\psi(|x-y|)} \cdot \frac{\varphi(|x-y|)}{|f(x)-f(y)|}$$

and

$$\frac{\psi(\delta)}{\varphi(1)} \cdot \frac{|f(x)-f(y)|}{\psi(|x-y|)} \leq \frac{|f(x)-f(y)|}{\varphi(|x-y|)}$$

for all $x, y \in K$ such that $|x-y| \geq \delta$. (In the case of $f(x) = f(y)$ the second inequalitiy holds trivially.) Finally

$$\frac{\psi(\delta)}{\varphi(1)} \cdot \sup_{\substack{x,y\in K\\ |x-y|\geq\delta}} \frac{|f(x)-f(y)|}{\psi(|x-y|)} \leq \sup_{\substack{x,y\in K\\ |x-y|\geq\delta}} \frac{|f(x)-f(y)|}{\varphi(|x-y|)}.$$

Similarly, we have from (3):

$$C_1 \,.\, \sup_{\substack{x,y\in K\\ |x-y|<\delta}} \frac{|f(x)-f(y)|}{\psi(|x-y|)} \leq \sup_{\substack{x,y\in K\\ |x-y|<\delta}} \frac{|f(x)-f(y)|}{\varphi(|x-y|)}.$$

Denote

$$a := \min\left\{C_1, \frac{\psi(\delta)}{\varphi(1)}\right\} > 0.$$

Then we have

$$|f|_\psi \leq \frac{1}{a}|f|_\varphi,$$

hence $C^{0,\psi}(K) \hookrightarrow C^{0,\varphi}(K)$.

(ii) Both φ and ψ are non-negative on K, hence $C_2 = \frac{1}{C_1}$. Therefore, (ii) is a direct consequence of (i).
(iii) From (i) and (ii), we obtain that $C^{0,\varphi}(K) \hookrightarrow C^{0,\psi}(K)$ as well as $C^{0,\psi}(K) \hookrightarrow C^{0,\varphi}(K)$, hence $C^{0,\varphi}(K) \cong C^{0,\psi}(K)$ by definition. $\square$

Remark 15. One notices that although we assumed $\varphi, \psi \in H$ in the previous theorem, it holds even for φ, ψ not satisfying (H3). Then $(C^{0,\varphi}(K), \|\cdot\|_\varphi)$ is still defined correctly though we do not call it "Hölder" in this case. This will be true also in some of the following statements and we will sometimes refer to them in the sense of their "non–H3" versions without further notice.

However, the reason to add the (H3)-condition is to establish a wider class of functions (of the Hölder type), as seen in the following corollary.

Corollary 16. *Let φ be a function satisfying the conditions* (H1), (H2) *and the modified condition*

(H3*) $\lim_{t\to 0+} \frac{\varphi(t)}{t} = C > 0$.

Then $C^{0,\varphi}(K) \cong \mathrm{Lip}(K)$.

Proof. Consider $\psi(t) := t$ and apply Theorem 14. (See Remark 15.) $\square$

Theorem 14 provides a sufficient condition for existence of continuous embedding of a Hölder space into another one. However, we would like to improve it to obtain a characterization of the existence of those embeddings. The folowing theorem tells us that if, in particular, $\varphi \in C^{0,\varphi}(K)$ (i.e., φ is Hölder in the norm generated by φ itself), then the condition introduced in Theorem 14 (i) is necessary, too.

Theorem 17. *Consider functions $\varphi, \psi \in H$ and denote*

$$C_1 := \liminf_{t\to 0+} \frac{\psi(t)}{\varphi(t)}, \quad C_2 := \limsup_{t\to 0+} \frac{\psi(t)}{\varphi(t)}.$$

Then the following statements hold:

(i) *If $\varphi \in C^{0,\varphi}(K)$, then $C^{0,\varphi}(K) \hookrightarrow C^{0,\psi}(K)$ if and only if $C_1 > 0$.*

(ii) *If $\psi \in C^{0,\psi}(K)$, then $C^{0,\psi}(K) \hookrightarrow C^{0,\varphi}(K)$ if and only if $C_2 < \infty$.*

(iii) *If $\varphi \in C^{0,\varphi}(K)$ and $\psi \in C^{0,\psi}(K)$, then $C^{0,\varphi}(K) \cong C^{0,\psi}(K)$ if and only if $0 < C_1 \le C_2 < \infty$.*

Proof. The "only if" parts of all three statements are proven by Theorem 14 under weaker assumptions.
We shall prove the "if" part of (i). The "if" parts of (ii) and (iii) will then follow obviously. So, suppose that $C_1 = 0$. We will show that then it even holds that $C^{0,\varphi}(K) \nsubseteq C^{0,\psi}(K)$.
To do so, we have to find a function $f \in C^{0,\varphi}(K) \setminus C^{0,\psi}(K)$. Under given conditions, it is not a very difficult task. We define f simply as

$$f(t) := \varphi(t), \quad t \in K.$$

By the initial assumption, there exists a decreasing sequence $\{t_n\}_{n=1}^{\infty} \subset (0,1)$ such that $t_n \searrow 0$ and

$$\frac{\varphi(t_n)}{\psi(t_n)} > n \quad \forall\, n \in \mathbb{N}. \tag{4}$$

Moreover, f $(\equiv \varphi)$ is φ-Hölder and it holds that

$$\frac{|f(t_n) - f(0)|}{\varphi(t_n)} = \frac{f(t_n)}{\varphi(t_n)} = 1 \quad \forall\, n \in \mathbb{N}.$$

(Remember that $f \equiv \varphi \in H$, hence $f(0) = 0$ and $f(t) > 0$ for $0 < t < 1$.)
On the other hand, from (4) we have

$$\frac{f(t_n)}{\psi(t_n)} > n \frac{f(t_n)}{\varphi(t_n)} = n \quad \forall\, n \in \mathbb{N},$$

therefore

$$|f|_{\psi} = \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|f(x) - f(y)|}{\psi(|x-y|)} \geq \sup_{n \in \mathbb{N}} \frac{|f(t_n) - f(0)|}{\psi(t_n)} = \sup_{n \in \mathbb{N}} \frac{f(t_n)}{\psi(t_n)} = \infty.$$

Thus, $f \in C^{0,\varphi}(K) \setminus C^{0,\psi}(K)$.

□

We can easily verify that the condition $\varphi \in C^{0,\varphi}(K)$ is satisfied, for instance, if the the function $\varphi \in H$ is non-convex:

Proposition 18. *Suppose that a function $\varphi \in H$ is non-convex on K. Then $\varphi \in C^{0,\varphi}(K)$.*

Proof. Since φ is non-decreasing and non-convex and $\varphi(0) = 0$, it holds that

$$|\varphi(x) - \varphi(y)| \leq \varphi(|x-y|) - \varphi(0) = \varphi(|x-y|),$$

thus $|\varphi|_{\varphi} < \infty$. □

Proposition 18 provides a particular class of functions meeting the condition $\varphi \in C^{0,\varphi}(K)$ but this generally does not hold for every function $\varphi \in H$. Thus, Theorem 17 cannot be used directly in a general case.

However, if there was a special function which both satisfied the problematic condition and had a "suitable" relation to the original fuction φ, maybe the theorem would be still useful. This idea proves to be right and it leads to the concept of the so-called "watch-functions" which we will introduce now.

Definition 19. Consider $t \in K$ and denote $\Lambda(t)$ the system of all finite divisions of t, which means

$$\Lambda(t) := \left\{ \{\lambda_i\}_{i=1}^{N\in\mathbb{N}} \colon \sum_{i=1}^{N} \lambda_i = t \right\}.$$

Then, for $\varphi \in H$ we define the *watch-function corresponding to* φ as

$$\widetilde{\varphi}(t) := \inf_{\Lambda(t)} \sum_{i=1}^{N} \varphi(\lambda_i), \quad t \in K.$$

To prevent confusion, we point out that N is not a fixed number but it depends on the particular division of t. Of course, different divisions of t generally have different numbers of their elements (which is expressed by N).

The watch-functions prove to have remarkable properties. Let us show them in the following theorem.

Theorem 20 (Properties of a watch-function)**.** *Consider a function $\varphi \in H$. The corresponding watch-function $\widetilde{\varphi}$ has following properties:*

(i) $\widetilde{\varphi}(t) \leq \varphi(t)$ *for all* $t \in K$*,*

(ii) $\widetilde{\varphi} \in H$*,*

(iii) $\widetilde{\varphi} \in C^{0,\widetilde{\varphi}}(K)$.

Proof. (i) Consider $t \in K$. The singleton $\{t\}$ is a trivial division of t, thus we have

$$\varphi(t) \geq \inf_{\Lambda(t)} \sum_{i=1}^{N} \varphi(\lambda_i) = \widetilde{\varphi}(t).$$

(ii) We have to verify that $\widetilde{\varphi}$ satisfies the conditions (H1)–(H3), provided that φ satisfies them.
Obviously, $\widetilde{\varphi}$ is a non-negative-valued function on K. To show that it is non-decreasing, choose $x, y \in K$ such that $x < y$ and let us prove that $\widetilde{\varphi}(x) \leq \widetilde{\varphi}(y)$.
Choose a division of y, i.e., the set $\{\lambda_i\}_{i=1}^N \in \Lambda(y)$. First, consider the case of $N \geq 2$. Then we can extract a subset $S := \{\lambda_{i_1}, \dots, \lambda_{i_r}\}$ such that

$$\sum_{j=1}^{r} \lambda_{i_j} < x \leq \sum_{j=1}^{r} \lambda_{i_j} + \lambda_k \quad \forall\, \lambda_k \notin S. \tag{5}$$

Notice that there can exist more subsets (and corresponding numbers r) satisfying (5). In this case, we just choose one of them. Anyway, it holds that $0 < r < N$. Define

$$\kappa_j := \begin{cases} \lambda_{i_j} & 1 \leq j \leq r \\ x - \sum_{j=1}^r \lambda_{i_j} & j = r+1. \end{cases}$$

If $N = 1$, then $\lambda_1 = y$ and we define $r := 0$, $\kappa_1 := x$ and $\kappa_2 := y - x$.
One sees that the set $S' := \{\kappa_1, \dots, \kappa_{r+1}\}$ is a division of x. Moreover, from (5) and the fact that φ is non-decreasing, it holds that

$$\sum_{j=1}^{r+1} \varphi(\kappa_j) \leq \sum_{i=1}^{N} \varphi(\lambda_i).$$

Obviously, the left side can be replaced by the infimum over all divisions $\{\mu_k\}_{k=1}^M \in \Lambda(x)$. On the right side, the passage to the infimum is also possible thanks to the arbitrariness of the division $\{\lambda\}_{i=1}^N \in \Lambda(y)$. Together, we have

$$\widetilde{\varphi}(x) = \inf_{\Lambda(x)} \sum_{k=1}^{M} \varphi(\mu_k) \leq \inf_{\Lambda(y)} \sum_{j=1}^{N} \varphi(\lambda_{i_j}).$$

Thus, we have just proven that $\widetilde{\varphi}$ is non-decreasing, hence it satisfies (H1).
(H2) is satisfied by $\widetilde{\varphi}$ as a consequence of (i).
To prove that $\widetilde{\varphi}$ satisfies also (H3), it suffices to show that for every $C \in \mathbb{R}$ there exists $\varepsilon > 0$ such that

$$\frac{\widetilde{\varphi}(t)}{t} \geq C \quad \forall\, t \in (0, \varepsilon).$$

Set $C \in \mathbb{R}$. We know that φ satisfies (H3), so there exists $\eta > 0$ such that

$$\frac{\varphi(t)}{t} \geq C \quad \forall\, t \in (0, \eta). \tag{6}$$

Define $\varepsilon := \eta$ and choose $t \in (0, \varepsilon)$. Using (6), for an arbitrary division $\{\lambda_i\}_{i=1}^N \in \Lambda(t)$, we have

$$\varphi(\lambda_i) \geq C\lambda_i \quad \forall\, i \in \{1, \dots, N\},$$

therefore

$$\sum_{i=1}^{N} \varphi(\lambda_i) \geq C \sum_{i=1}^{N} \lambda_i = t$$

and, by passing to the infmium:

$$\widetilde{\varphi}(t) \geq Ct.$$

Hence, $\widetilde{\varphi}$ satisfies (H3), thus $\widetilde{\varphi} \in H$.

(iii) We already have proven (ii), so $C^{0,\widetilde{\varphi}}(K)$ is defined correctly. We shall show that

$$|\widetilde{\varphi}(x) - \widetilde{\varphi}(y)| \leq \widetilde{\varphi}(|x-y|) \tag{7}$$

for all $x, y \in K$. Thus, choose (fixed) $x, y \in K$, without loss of generality assume that $x < y$. Then it holds by (i) that

$$|\widetilde{\varphi}(x) - \widetilde{\varphi}(y)| = \widetilde{\varphi}(y) - \widetilde{\varphi}(x).$$

By the definition of $\widetilde{\varphi}$, for every $\varepsilon > 0$ there exist divisions $\{\lambda_i\}_{i=1}^N \in \Lambda(x)$ and $\{\kappa_j\}_{j=1}^M \in \Lambda(y-x)$ such that

$$\sum_{i=1}^{N} \varphi(\lambda_i) \leq \widetilde{\varphi}(x) + \frac{\varepsilon}{2} \tag{8}$$

and

$$\sum_{j=1}^{M} \varphi(\kappa_j) \leq \widetilde{\varphi}(y-x) + \frac{\varepsilon}{2}. \tag{9}$$

Together, $\{\lambda_i\} \cup \{\kappa_j\} \in \Lambda(y)$, therefore it holds

$$\widetilde{\varphi}(y) \leq \sum_{i=1}^{N} \varphi(\lambda_i) + \sum_{j=1}^{M} \varphi(\kappa_j) \leq \widetilde{\varphi}(x) + \widetilde{\varphi}(y-x) + \varepsilon.$$

By the limit passage $\varepsilon \to 0+$ we have

$$\widetilde{\varphi}(y) \leq \widetilde{\varphi}(x) + \widetilde{\varphi}(y-x),$$

thus

$$\widetilde{\varphi}(y) - \widetilde{\varphi}(x) \leq \widetilde{\varphi}(y-x).$$

Since $x, y \in K$ were arbitrarily chosen, the inequality above holds for every pair $x, y \in K$, $x < y$, hence $\widetilde{\varphi} \in C^{0,\widetilde{\varphi}}(K)$. $\square$

Let us continue our investigation the relation of the spaces $C^{0,\varphi}(K)$ and $C^{0,\widetilde{\varphi}}(K)$. To be specific, we aim to prove that these spaces are identical. A key step on the path to proving that statement is the lemma below.

Lemma 21. *Suppose that $\varphi \in H$ and $f \in C^{0,\varphi}(K)$. Then we have*

$$|f(x) - f(y)| \leq |f|_{\varphi} . \widetilde{\varphi}(|x-y|) \quad \forall\ x, y \in K.$$

Proof. Choose $x, y \in K$, without loss of generality assume that $x \neq y$. (Otherwise, the requested inequality is trivial.) Let $\{\lambda_i\}_{i=1}^n \in \Lambda(|x-y|)$ be an arbitrary division of $|x-y|$.
Denote

$$x_i := \begin{cases} x & i = 0 \\ x + \sum_{j=1}^{i} \lambda_j & 1 \leq i \leq n-1 \\ y & i = n \end{cases}$$

We have

$$|f(x) - f(y)| \leq \sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})| \leq |f|_{\varphi} . \left(\sum_{i=0}^{n} \varphi(\lambda_i) \right).$$

Now, since the division $\{\lambda_i\}_{i=1}^n$ was arbitrarily chosen from $\Lambda(|x-y|)$, we can pass to the infimum over $\Lambda(|x-y|)$ on the right side, thus

$$|f(x) - f(y)| \leq |f|_{\varphi} . \inf_{\Lambda(|x-y|)} \sum_{i=1}^{n} \varphi(\lambda_i) = |f|_{\varphi} . \widetilde{\varphi}(|x-y|).$$

$\square$

Using the previous result, we are able to prove the following theorem which is of a strategic importance. It states that the spaces $C^{0,\varphi}(K)$ and $C^{0,\widetilde{\varphi}}(K)$ are equivalent even with unit constants of norms equivalence.

Theorem 22. *Let $f \colon K \to \mathbb{R}$ be a function and $\varphi \in H$. Then it holds that*

$$\|f\|_{\varphi} = \|f\|_{\widetilde{\varphi}}.$$

Proof. It suffices to prove that $|f|_{\varphi} = |f|_{\widetilde{\varphi}}$.
Let x, y be arbitrary points such that $x, y \in K$ and $x \neq y$. By Lemma 21, it holds that

$$\omega(f, |x-y|) \leq |f|_{\varphi} . \widetilde{\varphi}(|x-y|),$$

thus, in paticular,

$$\frac{|f(x)-f(y)|}{\widetilde{\varphi}(|x-y|)} \leq |f|_{\varphi}. \tag{10}$$

Next, by Theorem 20 (i) we have

$$\frac{|f(x)-f(y)|}{\varphi(|x-y|)} \leq \frac{|f(x)-f(y)|}{\widetilde{\varphi}(|x-y|)}. \tag{11}$$

Combining (10) and (11), we obtain

$$\frac{|f(x)-f(y)|}{\varphi(|x-y|)} \leq \frac{|f(x)-f(y)|}{\widetilde{\varphi}(|x-y|)} \leq |f|_{\varphi},$$

therefore, thanks to the choice of x, y:

$$\sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x)-f(y)|}{\varphi(|x-y|)} \leq \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x)-f(y)|}{\widetilde{\varphi}|x-y|} \leq |f|_{\varphi},$$

thus $|f|_{\varphi} \leq |f|_{\widetilde{\varphi}} \leq |f|_{\varphi}$, which means that $\|f\|_{\varphi} = \|f\|_{\widetilde{\varphi}}$. By the proof it is also clear that $\|f\|_{\varphi} = \infty$ if and only if $\|f\|_{\widetilde{\varphi}} = \infty$, and in this case the statement holds trivially. $\square$

At last, we can now formulate a necessary and sufficient condition of the existence of continuous embedding of Hölder spaces.

Theorem 23 (A characterization of the existence of a continuous embedding of Hölder spaces)**.** *Consider two functions $\varphi, \psi \in H$. Denote*

$$C_1 := \liminf_{t\to 0+} \frac{\psi(t)}{\widetilde{\varphi}(t)}, \quad C_2 := \limsup_{t\to 0+} \frac{\widetilde{\psi}(t)}{\varphi(t)}.$$

Then the following statements hold:

(i) *If $\varphi \in C^{0,\varphi}(K)$, then $C^{0,\varphi}(K) \hookrightarrow C^{0,\psi}(K)$ if and only if $C_1 > 0$.*

(ii) *If $\psi \in C^{0,\psi}(K)$, then $C^{0,\psi}(K) \hookrightarrow C^{0,\varphi}(K)$ if and only if $C_2 < \infty$.*

(iii) *If $\varphi \in C^{0,\varphi}(K)$ and $\psi \in C^{0,\psi}(K)$, then $C^{0,\varphi}(K) \cong C^{0,\psi}(K)$ if and only if $0 < C_1 \leq C_2 < \infty$.*

Proof. (i). By Theorem 22, it holds that $C^{0,\varphi}(K) \hookrightarrow C^{0,\psi}(K)$ if and only if

$$C^{0,\widetilde{\varphi}}(K) \hookrightarrow C^{0,\psi}(K). \tag{12}$$

From Theorem 20 (iii) and Theorem 17, (12) is true if and only if $C_1 > 0$. Statements (ii) and (iii) can be proven similarly. $\square$

4 Compact embeddings of the Hölder spaces

In a similar way as before, we will now study the compact embeddings of Hölder spaces. At first, the completeness of the Hölder spaces needs to be proven.

Theorem 24. *Assume that $\varphi \in H$. $(C^{0,\varphi}(K), \|\cdot\|_\varphi)$ is then a Banach space.*

Proof. Clearly, $(C^{0,\varphi}(K), \|\cdot\|_\varphi)$ is a normed linear space. We have to show that it is complete.
Consider a Cauchy sequence $\{f_n\}_{n=1}^\infty$ in $(C^{0,\varphi}(K), \|\cdot\|_\varphi)$.
Thanks to (1), $\{f_n\}$ is a Cauchy sequence in $\|\cdot\|$ (as well as in $|\cdot|_\varphi$). Since $f_n \in C(K)$ for all $n \in \mathbb{N}$ and $(C(K), \|\cdot\|)$ is complete, there exists $f \in C(K)$ such that $f_n \rightrightarrows f$ (on K).
Being Cauchy in $|\cdot|_\varphi$, $\{f_n\}$ is necessarily bounded in this seminorm, thus there is a constant $P \in \mathbb{R}$ such that

$$\frac{|f_n(x) - f_n(y)|}{\varphi(|x-y|)} < P \quad \forall\, n \in \mathbb{N},\ \forall\, x, y \in K,\ x \neq y.$$

f is the uniform limit of f_n, so by the limit passage we have

$$\frac{|f(x) - f(y)|}{\varphi(|x-y|)} = \lim_{n\to\infty} \frac{|f_n(x) - f_n(y)|}{\varphi(|x-y|)} \leq P \quad \forall\, x, y \in K,\ x \neq y$$

in other words $f \in C^{0,\varphi}(K)$.
It remains to prove that f_n converges to f in the φ-Hölder seminorm.
Set $\varepsilon > 0$. $\{f_n\}$ is Cauchy in that seminorm, hence there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f_n(x) - f_m(x) - f_n(y) + f_m(y)|}{\varphi(|x-y|)} < \varepsilon \quad \forall\, m, n \in \mathbb{N},\ m, n > n_0.$$

Using the limit passage $m \to \infty$ we have

$$\sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f_n(x) - f(x) - f_n(y) + f(y)|}{\varphi(|x-y|)} \leq \varepsilon \quad \forall\, n \in \mathbb{N},\ n > n_0.$$

Since ε was arbitrarily chosen, $|f_n - f|_\varphi \to 0$, the Cauchy sequence converges (in $\|\cdot\|_\varphi$), hence the space $(C^{0,\varphi}(K), \|\cdot\|_\varphi)$ is complete and so is the proof. □

Definition 25. For $\varphi \in H$ we denote

$$B_\varphi := \{f \in C^{0,\varphi}(K) \colon \|f\|_\varphi \le 1\},$$

the unit ball in $C^{0,\varphi}(K)$.

At this place let us just notice that when we need to prove that a space $(X, \|\cdot\|_X)$ is compactly embedded into $(Y, \|\cdot\|_Y)$, we do so by using the definition, so we verify that the unit ball in X is relatively compact in Y.

Lemma 26. *Suppose that $\varphi \in H$. Then $C^{0,\varphi}(K) \hookrightarrow\hookrightarrow C(K)$.*

Proof. $\|f\| + |f|_\varphi = \|f\|_\varphi \le 1$ for all $f \in B_\varphi$, so B_φ is bounded in $\|\cdot\|$. Choose $\varepsilon > 0$. From (H2), we can find $\delta > 0$ such that

$$\varphi(|x-y|) < \varepsilon \quad \forall\, x, y \in K,\, |x-y| < \delta.$$

Since

$$|f(x) - f(y)| \le \varphi(|x-y|) \quad \forall\, f \in \bar{B}_\varphi, \forall\, x, y \in K,$$

we have

$$|f(x) - f(y)| < \varepsilon \quad \forall\, f \in \bar{B}_\varphi, \forall\, x, y \in K,\, |x-y| < \delta.$$

Thus, B_φ is equicontinuous, hence, by Arzelà-Ascoli Theorem, it is relatively compact in $(C(K), \|\cdot\|)$. $\square$

The next result will be useful for the proof of the main theorems dealing with compact embeddings of Hölder spaces.

Lemma 27. *Suppose $\varphi, \psi \in H$ and it holds that*

$$\lim_{t \to 0+} \frac{\psi(t)}{\varphi(t)} = 0. \tag{13}$$

Then, for every $\varepsilon > 0$, there exists a $\delta \in (0,1)$ such that, for all $f \in C(K)$:

$$|f|_\varphi \le \max\left\{\varepsilon\, |f|_\psi, \frac{2\,\|f\|}{\varphi(\delta)}\right\}.$$

Proof. Set $\varepsilon > 0$. Since (13) holds, there exists $\delta \in (0,1)$ such that

$$\frac{\psi(t)}{\varphi(t)} < \varepsilon \quad \forall\, t \in (0, \delta). \tag{14}$$

We have

$$
\begin{aligned}
|f|_\varphi &= \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|f(x)-f(y)|}{\varphi(|x-y|)} = \\
&= \max\left\{ \sup_{\substack{x,y\in K\\ 0<|x-y|<\delta}} \frac{|f(x)-f(y)|}{\varphi(|x-y|)},\ \sup_{\substack{x,y\in K\\ \delta\le|x-y|\le 1}} \frac{|f(x)-f(y)|}{\varphi(|x-y|)} \right\} \le \\
&\le \max\left\{ \sup_{\substack{x,y\in K\\ 0<|x-y|<\delta}} \left(\frac{\psi(|x-y|)}{\varphi(|x-y|)}\cdot\frac{|f(x)-f(y)|}{\psi(|x-y|)}\right),\ \sup_{\substack{x,y\in K\\ \delta\le|x-y|\le 1}} \frac{|f(x)-f(y)|}{\varphi(\delta)} \right\} \le \\
&\le \max\left\{ \varepsilon\,|f|_\psi, \frac{2\,\|f\|}{\varphi(\delta)} \right\}.
\end{aligned}
$$

(We used (14) and the fact that φ is increasing.) $\square$

In the theorems involving continuous embeddings we needed some limit superior to be a positive real number. If we want the embedding to be even compact, the limit superior has to be zero, as we will see in the following.

Theorem 28. *Consider functions $\varphi, \psi \in H$.*

(i) *If it holds that*

$$\lim_{t\to 0+} \frac{\psi(t)}{\varphi(t)} = 0, \tag{15}$$

then $C^{0,\psi}(K) \hookrightarrow\hookrightarrow C^{0,\varphi}(K)$.

(ii) *If, moreover, $\psi \in C^{0,\psi}(K)$, then $C^{0,\psi}(K) \hookrightarrow\hookrightarrow C^{0,\varphi}(K)$ if and only if (15) holds.*

Proof. (i) As it has been said above, we will prove that $B_\psi \subset C^{0,\psi}(K)$ is a relatively compact set in $C^{0,\varphi}(K)$. So, consider an arbitrary sequence $\{f_n\}_{n=1}^\infty \subset B_\psi$. From Lemma 26, we know that $\{f_n\}$ has a subsequence converging in $\|\cdot\|$. Without loss of generality, we assume that it is $\{f_n\}$ itself. It means that there exists a $f \in C(K)$ such that $f_n \rightrightarrows f$ on K (because $C(K)$ is a Banach space). Of course, $\{f_n\}$ is then uniformly Cauchy (i.e., Cauchy in $\|\cdot\|$). We shall show that $\{f_n\}$ is Cauchy also in $|\cdot|_\varphi$.
Since $f_n \in B_\psi$ for all $n \in \mathbb{N}$, it holds for all $m, n \in \mathbb{N}$ that

$$\|f_n - f_m\| \le 2 \quad\text{and}\quad |f_n - f_m|_\psi \le 2. \tag{16}$$

Set $\varepsilon > 0$. By Lemma 27 there exists $\delta \in (0,1)$ such that

$$|f_n - f_m|_\varphi \le \max\left\{\frac{\varepsilon}{4}\,|f_n - f_m|_\psi, \frac{2\,\|f_n - f_m\|}{\varphi(\delta)}\right\} \quad \forall\, m,n \in \mathbb{N}. \tag{17}$$

Now, $\|f_n\|$ is uniformly Cauchy, hence there exists $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\| \le \frac{\varepsilon\varphi(\delta)}{4} \quad m,n \in \mathbb{N},\ m,n > n_0.$$

Therefore, using (16) and (17) we have for all $m,n \in \mathbb{N}$, $m,n > n_0$:

$$|f_n - f_m|_\varphi \le \max\left\{\frac{\varepsilon}{4}\,|f_n - f_m|_\psi, \frac{2\,\|f_n - f_m\|}{\varphi(\delta)}\right\} \le \frac{\varepsilon}{2} < \varepsilon.$$

Hence, $\{f_n\}$ is Cauchy in $|\cdot|_\varphi$. Thus, it is Cauchy also in $\|\cdot\|_\varphi = \|\cdot\| + |\cdot|_\varphi$ and, by the completeness of $C^{0,\varphi}(K)$ (Theorem 24), $\{f_n\}$ converges in $C^{0,\varphi}(K)$. Therefore, B_ψ is relatively compact in $C^{0,\varphi}(K)$.
(ii) To prove necessity of (15), let us assume that $\psi \in C^{0,\psi}(K)$ but (15) does not hold. Then we will show that B_ψ is not relatively compact in $C^{0,\varphi}(K)$ by constructing a sequence in B_ψ which has no subseqence converging in $|\cdot|_\varphi$.
Since (15) does not hold, there exists $\varepsilon > 0$ and a decreasing sequence $\{t_n\}_{n=1}^\infty \subset (0,1)$ such that $t_n \searrow 0$ and

$$\frac{\psi(t_n)}{\varphi(t_n)} > \varepsilon \quad \forall\, n \in \mathbb{N}. \tag{18}$$

Without loss of generality we assume that

$$t_n < 2^{-n} \quad \forall\, n \in \mathbb{N}$$

and

$$\psi(t_n) < 1 \quad \forall\, n \in \mathbb{N}.$$

We know that $\psi \in C^{0,\psi}(K)$, thus $\|\psi\|_\psi < \infty$. Now define the sequence of functions $\{f_n\}_{n=1}^\infty$ as

$$f_n(t) := \begin{cases} 0 & 0 \le t < 2^{-n} \\ \dfrac{\psi(t - 2^{-n})}{2\|\psi\|_\psi} & 2^{-n} \le t < 2^{-n} + t_n \\ \dfrac{\psi(t_n)}{2\|\psi\|_\psi} & 2^{-n} + t_n \le t \le 1. \end{cases}$$

It is easy to verify that $\{f_n\} \subset B_\psi$. On the other hand, according to (18) and the construction of the sequence, we have

$$|f_m - f_n|_\varphi > \varepsilon \quad \forall\, m, n \in \mathbb{N},\, m \neq n,$$

therefore no subsequence of $\{f_n\}$ can converge in $\|\cdot\|_\varphi$, hence B_ψ is not relatively compact in $C^{0,\varphi}(K)$ and the proof is finished. □

Thanks to the results above we can now state the main theorem.

Theorem 29 (A characterization of existence of compact embeddings of Hölder spaces)**.** *Consider functions $\varphi, \psi \in H$ The following conditions are equivalent:*

(i) $C^{0,\psi}(K) \hookrightarrow\hookrightarrow C^{0,\varphi}(K)$,

(ii) $\displaystyle\lim_{t\to 0+} \frac{\widetilde{\psi}(t)}{\varphi(t)} = 0.$

Proof. By Theorem 22, (i) holds if and only if $C^{0,\widetilde{\psi}}(K) \hookrightarrow\hookrightarrow C^{0,\varphi}(K)$ and, according to Theorem 20 (iii) and Theorem 28, this is equivalent to (ii). □

In Lemma 26 we showed that $C^{0,\varphi}(K) \hookrightarrow\hookrightarrow C(K)$. Similarly we can ask if it holds that $C^1(K) \hookrightarrow\hookrightarrow C^{0,\varphi}(K)$ for instance. The answer is positive and not hard to prove.

Lemma 30. *For any $\varphi \in H$ it holds that*

$$\mathrm{Lip}(K) \hookrightarrow\hookrightarrow C^{0,\varphi}(K) \quad \text{and} \quad C^1(K) \hookrightarrow\hookrightarrow C^{0,\varphi}(K).$$

Proof. Obviously $C^1(K) \subset \mathrm{Lip}(K)$, thus consider $\psi(t) = t$ and apply Theorem 28. (See Remark 15.) □

Our last results are connected to the spaces of functions on K whose (first or higher) derivatives are Hölder functions. At least, it also explains why the "zero" symbol is used in the notation $C^{0,\varphi}(K)$.

Definition 31. Let n be a positive integer and $\varphi \in H$. We define the space $C^{n,\varphi}(K)$ as

$$C^{n,\varphi}(K) := \{f \in C^n(K) \colon f^{(n)} \in C^{0,\varphi}(K)\}.$$

The norm of this space is defined as

$$\|\cdot\|_{n,\varphi} := \|\cdot\|_n + |\cdot|_\varphi$$

Theorem 32. *Suppose that $\varphi, \psi \in H$ and m, n are non-negative integers. Then $C^{n,\psi} \hookrightarrow\hookrightarrow C^{m,\varphi}$ if and only if one of the following conditions is satisfied:*

(i) $m < n$,

(ii) $m = n$ *and* $\lim_{t\to 0+} \frac{\widetilde{\psi}(t)}{\varphi(t)} = 0$.

Proof. The (i) part follows the fact that $C^{n,\psi} \hookrightarrow\hookrightarrow C^{m,\varphi}$ if $m < n$. This can be proven easily using the Arzelà-Ascoli theorem. The second part is a consequence of Lemmas 26, 30 and Theorem 29. $\square$

As a last remark, we will show one particular example in which we apply the previous results.

We have adopted a general approach employing the function $\varphi \in H$ in the definition of the Hölder fuctions. However, a narrower definition is also quite common, using $\varphi(t) := t^\alpha$ with $\alpha \in (0, 1)$ to establish the Hölder class of functions.
Let us now focus on this particular case of Hölder functions a bit more.
For short we will denote by $C^{0,\alpha}(K)$ the space of α-Hölder functions, i.e., the φ-Hölder ones for which $\varphi(t) := t^\alpha$ $(\alpha \in (0, 1))$. The norm $\| \cdot \|_\alpha$ is defined similarly.
Consider $\alpha, \beta \in (0, 1)$, non-negative integers m, n such that $m + \alpha < n + \beta$ and the spaces $C^{m,\alpha}(K)$, $C^{n,\beta}(K)$ equipped with the norms $\| \cdot \|_{m,\alpha}$ and $\| \cdot \|_{n,\beta}$ respectively. ($C^{m,\alpha}(K) := C^{m,\varphi}(K), \varphi(t) := t^\alpha$.)
From Theorem 32 we know that $C^{n,\beta} \hookrightarrow\hookrightarrow C^{m,\alpha}$.
Now, for $\gamma \geq 0$ and $k = \max\{n \in \mathbb{N} \cup \{0\} : n \leq \gamma\}$ denote shortly

$$C^\gamma := \begin{cases} C^k & \gamma = k \\ C^{k,\gamma-k} & \text{else.} \end{cases}$$

Notice that we have just shown a possibility how to extend the (countable) sequence of spaces $\{C^0(K), C^1(K), C^2(K), \ldots\}$ into the (uncountable) sequence $\{C^\gamma(K)\}_{\gamma\geq 0}$ while preserving the property $C^{\gamma_1}(K) \hookrightarrow\hookrightarrow C^{\gamma_2}(K)$ if $\gamma_1 < \gamma_2$.

References

[1] F. Hirsch, G. Lacombe, *Elements of Functional Analysis*, Springer, New York, 1997