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σ-porosity and Infinite Games

Contents

Český abstrakt: Charakterizujeme σ-P-pórovité množiny v libovolném úplném metrickém prostoru pomocí nekonečné hry, kde P je libovolná relace pórovitosti. Tato charakterizace může být použita zejména pro případ obyčejné pórovitosti, ale také pro mnoho jiných variant pórovitosti.
Klíčová slova: nekonečné hry, pórovitost

English abstract: We characterize σ-P-porous sets in any complete metric space via an infinite game where P is an arbitrary porosity-like relation. This can be applied to ordinary porosity above all but also for many other variants of porosity.
Keywords: infinite games, porosity

1. Introduction.

The theory of porous and σ-porous sets forms an important part of real analysis and Banach space theory for more than forty years. It originated in 1967 when E. P. Dolženko used for the first time the nomenclature 'porous set' and proved that some sets of his interest are σ-porous (see [1]). Since then the porosity has been used many times especially in the differentiation theory (see [4] for an example). A very useful fact is that every σ-porous set (in $\mathbb{R}^n$) is of the first category and has Lebesgue measure zero. In many cases, it is much more comfortable to prove that a given set is σ-porous than proving that the set is both small in the sense of category and in the sense of measure. On the other hand, not every set of the first category and measure zero is also σ-porous which was first proved by L. Zajíček in [6] (although E. P. Dolženko stated this assertion without proof earlier).

The main aim of my work is finding an infinite game which characterizes σ-porous sets in as much general metric spaces as possible. This characterization should be similar to the very well known characterization of meager sets using so called Banach-Mazur game. This means, I would like to find an infinite game such that a set A is σ-porous if and only if the second player has a winning strategy in this game (which depends on the set A). A connection between σ-porosity and infinite games was first shown by M. Zelený in [9] where a sufficient condition for σ-porosity is given via an infinite game (but more complicated than just having a winning strategy). In Cantor topological space (where porosity is defined in a very natural way), the characterization was found by J. Zapletal and is described in a joint paper of J. Zapletal and I. Farah (see [3]). This was generalized by D. Rojas-Rebolledo, who found a similar characterization of σ-porosity (and also of σ-strong porosity) in any zero-dimensional metric space (see [5]). However, both these cases concerns only very special cases. In my work, I describe an infinite game which characterizes σ-P-porous sets in any complete metric space X where P is an arbitrary porosity-like relation on X.

2. Preliminaries.

Let (X, d) be a metric space. An open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. If $A \subseteq X$ then $\operatorname{diam} A = \sup\{d(a, b) \colon a \in A, b \in A\}$.

We will prove our result for a general porosity-like relation. To do this, we need the following definition.

Definition 2.1. Let X be a metric space and let $P \subseteq X \times 2^X$ be a relation between points of X and subsets of X. Then P is called a *point-set relation on* X. The symbol $P(x, A)$ where $x \in X$ and $A \subseteq X$ means that $(x, A) \in P$. For $A \subseteq X$ and $B \subseteq X$, we also use the symbol $P(A, B)$ which is equivalent to $[P(a, B)$ for every $a \in A]$.

The point-set relation P on X is called a *porosity-like relation* if moreover the following conditions hold:

(P1) if $A \subseteq B \subseteq X$, $x \in X$ and $P(x, B)$ then also $P(x, A)$,
(P2) if $A \subseteq X$ and $x \in X$ then $P(x, A)$ if and only if there exists $r > 0$ such that $P(x, A \cap B(x, r))$,
(P3) if $A \subseteq X$ and $x \in X$ then $P(x, A)$ if and only if $P\left(x, \overline{A}\right)$.

If P is a porosity-like relation on X, $A \subseteq X$ and $x \in X$, we say that

- A is *P-porous at x* if $P(x, A)$,
- A is *P-porous* if it is P-porous at every its point,
- A is *σ-P-porous* if it is a countable union of P-porous sets.

It can be easily checked that almost all commonly used variants of porosities (understood as point-set relations in a natural way) are porosity-like relations. Therefore, the characterization from the next chapter can be applied to all of them, namely to ordinary porosity (see the following definition), strong porosity (see [7, p. 317]), symmetrical porosity (see [7, p. 320]), but also right and left porosity (see [7, p. 317]), g-porosity (see [10, p. 35]), etc. To give an example, here is the definition of ordinary porosity.

Definition 2.2. Let (X, d) be a metric space. Let $A \subseteq X$, $x \in X$ and $R > 0$. We denote

$$\begin{aligned}\gamma(x, R, A) = \sup\bigl\{r > 0 \colon & \text{ there exists } z \in X \text{ such that} \\ & d(x, z) < R \text{ and } B(z, r) \cap A = \emptyset\bigr\},\end{aligned}$$

$$p(x, A) = \limsup_{R \to 0+} \frac{\gamma(x, R, A)}{R}.$$

We say that

- A is *(ordinary) porous at x* if $p(x, A) > 0$,
- A is *(ordinary) porous* if it is porous at every its point,
- A is *σ-(ordinary) porous* if it is a countable union of porous sets.

We will need the following theorem which can be found in [8, Lemma 3].

Theorem 2.3 ([8, Lemma 3])**.** *Let X be a metric space, P be a porosity-like relation on X and $A \subseteq X$. Then A is σ-P-porous if and only if for every $x \in A$ there exists $r > 0$ such that $B(x, r) \cap A$ is σ-P-porous.*

It is also necessary to remind some basic definitions which concern infinite games. Let A be a nonempty set and $n \in \mathbb{N}$. We denote by A^n the set of all sequences $s = (s_0, s_1, \ldots, s_{n-1})$ of length n from A. We also set $A^0 = \{\emptyset\}$ where $\emptyset$ is the *empty sequence* (of length 0). We denote by $A^{<\mathbb{N}}$ (resp. $A^{\mathbb{N}}$) the set of all finite (resp. infinite) sequences from A. This means that

$$A^{<\mathbb{N}} = \bigcup_{n=0}^{\infty} A^n.$$

The *length of a finite sequence* s is denoted by $\mathit{length}(s)$. If $s \in A^{<\mathbb{N}}$ and $n \leq \mathit{length}(s)$ then $s|n = (s_0, s_1, \ldots, s_{n-1}) \in A^n$. If $s, t \in A^{<\mathbb{N}}$ then we say that s is an *initial segment* of t and t is an *extension* of s if there exists $n \in \mathbb{N}$ such that $n \leq \mathit{length}(t)$ and $s = t|n$. If $s = (s_0, s_1, \ldots, s_{n-1}) \in A^n$ and $t = (t_0, t_1, \ldots, t_{m-1}) \in A^m$, then the *concatenation* of s and t is the sequence $s^\wedge t = (s_0, s_1, \ldots, s_{n-1}, t_0, t_1, \ldots, t_{m-1}) \in A^{n+m}$. If $x = (x_n)_{n=1}^{\infty} \in A^{\mathbb{N}}$ and $n \in \mathbb{N}$ then $x|n = (x_0, x_1, \ldots, x_{n-1}) \in A^n$. A finite sequence $s \in A^{<\mathbb{N}}$ is an *initial segment* of $x \in A^{\mathbb{N}}$ if $s = x|n$ for some $n \in \mathbb{N}$.

A subset $T \subseteq A^{<\mathbb{N}}$ is called a *tree* on A if for every $t \in T$ and every initial segment s of t, we have $s \in T$. A sequence $x \in A^{\mathbb{N}}$ is called an *infinite branch* of T if $x|n \in T$ for every $n \in \mathbb{N}$. The *body* of T is the set of all infinite branches of T and is denoted by $[T]$. This means that

$$[T] = \{x \in A^{\mathbb{N}} \colon x|n \in T \text{ for every } n \in \mathbb{N}\}.$$

A tree T is called *pruned* if every $s \in T$ has a proper extension in T, i.e. for every $s \in T$ there exists $t \in T$ such that t is an extension of s and $t \neq s$.

Let A be a nonempty set and $X \subseteq A^{\mathbb{N}}$. We associate X (which is called a *payoff set* then) with the following game:

I	a_0		a_2		a_4		$\ldots$
II		a_1		a_3		a_5	

Player I plays $a_0 \in A$, then player II plays $a_1 \in A$, I plays $a_2 \in A$, etc. Player I wins if $(a_n)_{n=1}^{\infty} \in X$, II wins in the opposite case. We denote this game by $G(A, X)$.

A *strategy* for player I in the game $G(A, X)$ is a tree $\sigma \subseteq A^{<\mathbb{N}}$ on A such that

- σ is nonempty,

- if $i \in \mathbb{N} \cup \{0\}$ and $(a_0, a_1, \dots, a_{2i}) \in \sigma$ then also $(a_0, a_1, \dots, a_{2i}, a_{2i+1}) \in \sigma$ for every $a_{2i+1} \in A$,
- if $i \in \mathbb{N} \cup \{0\}$ and $(a_0, a_1, \dots, a_{2i-1}) \in \sigma$ then there exists a unique $a_{2i} \in A$ such that $(a_0, a_1, \dots, a_{2i-1}, a_{2i}) \in \sigma$.

If we say that player I follows the strategy σ, we mean the following. Player I starts with the unique $a_0 \in A$ such that $(a_0) \in \sigma$. If II replies by $a_1 \in A$ then $(a_0, a_1) \in \sigma$ and I plays the unique $a_2 \in A$ such that $(a_0, a_1, a_2) \in \sigma$, etc.

A strategy for player I is *winning* in the game $G(A, X)$ if for every run $(a_n)_{n=1}^{\infty} \in A^{\mathbb{N}}$ of the game, in which I follows the strategy, $(a_n)_{n=1}^{\infty} \in X$ (and so I wins the run).

In a similar way, we define a (winning) strategy for II.

In the game $G(A, X)$, both players play arbitrary elements from a given nonempty set A. In many cases (also in this work), it is more convenient to let them obey some rules which are represented by a nonempty pruned tree $T \subseteq A^{<\mathbb{N}}$ (which determines so called *legal positions*). Let $X \subseteq [T]$ (X is called a *payoff set* again), then we define the game $G(T, X)$ as follows:

$$
\begin{array}{lcccccccc}
\text{I} & a_0 & & a_2 & & a_4 & & & \\
 & & & & & & & \dots & \\
\text{II} & & a_1 & & a_3 & & a_5 & &
\end{array}
$$

Again, I plays $a_0 \in A$, II plays $a_1 \in A$, etc. but both players have now to choose their moves such that $(a_0, a_1, \dots, a_n) \in T$ for every $n \in \mathbb{N} \cup \{0\}$. Player I wins if $(a_n)_{n=1}^{\infty} \in X$, II wins in the opposite case. The notion of (winning) strategy is defined analogously as before. However, this is only a special case of the previous game. Indeed, it is easy to see that if we denote

$$
\begin{aligned}
X' = \{x \in A^{\mathbb{N}} \colon \big(&\text{there exists } n \in \mathbb{N} \text{ such that } x|n \notin T \\
&\text{and the smallest such } n \text{ is even}\big) \text{ or } \ (x \in X)\},
\end{aligned}
$$

then I (resp. II) has a winning strategy in the game $G(T, X)$ if and only if I (resp. II) has a winning strategy in the game $G(A, X')$.

Finally, we will need the definition of a σ-discrete system of sets.

Definition 2.4. Let X be a topological space. A system $\mathcal{V}$ of subsets of X is said to be

- *discrete* if for every $x \in X$ there exists a neighborhood of x which intersects at most one set from the system $\mathcal{V}$,
- *σ-discrete* if it is a countable union of discrete systems.

We will use the existence of a σ-discrete basis of open sets in a metric space. This is guaranteed by the following theorem (proof can be found in [2, Theorem 4.4.3]).

Theorem 2.5 ([2, Theorem 4.4.3])**.** *Let X be a metrizable topological space. Then X has an open basis which is σ-discrete.*

3. Characterization of σ-P-porous sets in a complete metric space.

Let (X,d) be a nonempty complete metric space and $A\subseteq X$. Let P be a porosity-like relation on X. We define a game $G(A)$ between Boulder and Sisyfos as follows:

Boulder	B_1		B_2		B_3		$\dots$
Sisyfos		(S_1^1)		(S_2^1,S_2^2)		(S_3^1,S_3^2,S_3^3)	

(By using the names Boulder and Sisyfos, we follow the original terminology of J. Zapletal.) On the first move, Boulder plays an open ball $B_1\subseteq X$ and Sisyfos plays an open set $S_1^1\subseteq B_1$. On the second move, Boulder plays an open ball B_2 such that $\overline{B}_2\subseteq B_1$ and diam $B_2\le\frac{1}{2}$diam B_1 and Sisyfos plays open sets $S_2^1\subseteq B_2$ and $S_2^2\subseteq B_2$. On the nth move, Boulder plays an open ball B_n such that $\overline{B}_n\subseteq B_{n-1}$ and diam $B_n\le\frac{1}{2}$diam B_{n-1} and Sisyfos plays open sets $S_n^1\subseteq B_n, S_n^2\subseteq B_n,\dots,S_n^n\subseteq B_n$. After a run of the game $G(A)$, we get a unique point x lying in the intersection of the balls B_n, $n\in\mathbb{N}$. We call this point an *outcome* of the run. Sisyfos wins the run if at least one of the following conditions is satisfied:

(i) $x\notin A$,

(ii) there exists $m\in\mathbb{N}$ such that $x\in\ X\setminus\bigcup\limits_{n=m}^{\infty}S_n^m$ and $P\left(x,X\setminus\bigcup\limits_{n=m}^{\infty}S_n^m\right)$ (in this case, every such m is called a *witness of Sisyfos' victory*).

Boulder wins in the opposite case.

We say that a finite (also empty) sequence of open balls $(B_1,B_2,\dots,B_i)$ is *good* if the rules of the game $G(A)$ allow Boulder to play the ball B_n on his nth move, $n=1,2,\dots,i$. (In the game $G(A)$, this is independent of Sisyfos' moves.) Let σ be a strategy for Sisyfos in the game $G(A)$. Let us denote $B_0=X$. For $m\in\mathbb{N}\cup\{0\}$ and a good sequence $T=(B_1,B_2,\dots,B_i)$ we denote by M_m^T the set of all $x\in A\cap B_i$ such that in every run satisfying

($\star^T$) Boulder played the balls $B_1,B_2,\dots,B_i$ in sequence on his first i moves and Sisyfos kept on the strategy σ for the whole run,

and giving x as its outcome, all the witnesses of Sisyfos' victory (if there exist any) are greater than m. (The set M_m^T depends on the set A and on the strategy σ. This will not cause any difficulties since if we talk about this set later, both A and σ are always fixed.)

Let Boulder and Sisyfos play a run of the game $G(A)$. Let

$$V=(B_1,\mathcal{S}_1,B_2,\mathcal{S}_2,\dots),$$

$\mathcal{S}_n = (S_n^1, S_n^2, \dots, S_n^n)$, $n \in \mathbb{N}$, where Boulder played the ball B_n and Sisyfos played the sets $S_n^1, S_n^2, \dots, S_n^n$ on the nth move of the run, $n \in \mathbb{N}$. Then we will refer to the run itself by V and if we talk about the ball B_n or about the set S_n^m, $m \in \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, from the run V, we just use the symbols $B_n(V)$ and $S_n^m(V)$, respectively.

First of all, we prove the following lemma which is well known at least for ordinary porosity.

Lemma 3.1. *Let $\mathcal{V}$ be a σ-discrete system of σ-P-porous sets in X. Then $\bigcup \mathcal{V}$ is also σ-P-porous.*

Proof. Let $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ where $\mathcal{V}_n$ is a discrete system for every $n \in \mathbb{N}$. Let us take $n \in \mathbb{N}$ and $x \in X$. There exists $r > 0$ such that $B(x, r)$ intersects at most one set from the system $\mathcal{V}_n$. Therefore $B(x, r) \cap \bigcup \mathcal{V}_n$ is a σ-P-porous set. By Theorem 2.3, the set $\bigcup \mathcal{V}_n$ is σ-P-porous. Finally,

$$\bigcup \mathcal{V} = \bigcup_{n=1}^{\infty} \bigcup \mathcal{V}_n$$

is σ-P-porous as well. $\square$

The next technical lemma will be used to prove Theorem 3.3 which is our main result.

Lemma 3.2. *Let σ be a strategy for Sisyfos in the game $G(A)$. Let*

$$T_0 = (B_1, B_2, \dots, B_i)$$

be a good sequence of open balls and let $m \in \mathbb{N} \cup \{0\}$. Then there exist a P-porous set $N_m^{T_0}$ and a σ-discrete system $\mathcal{E}$ of sets such that

$$M_m^{T_0} = N_m^{T_0} \cup \bigcup \mathcal{E}$$

and, for every $E \in \mathcal{E}$, there exists a finite sequence T of open balls such that $T_0{}^\wedge T$ is good and $E \subseteq M_{m+1}^{T_0{}^\wedge T}$.

Proof. Denote

$$Z = \bigcup \left\{ S_n^{m+1}(V) : n \geq m+1, V \text{ is a run of the game } G(A) \text{ satisfying } (\star^{T_0}) \right\}.$$

Let us take $x \in Z$. Then we can find (and fix) $n(x) \geq m+1$ and a run $V(x)$ of the game $G(A)$ satisfying $(\star^{T_0})$ such that x lies in the open set $S_{n(x)}^{m+1}(V(x))$. Thus, if $\mathcal{B}$ is a σ-discrete basis of open sets in X (whose existence is guaranteed by Theorem 2.5) then the system

$$\mathcal{E}' = \left\{ B \in \mathcal{B} \colon B \subseteq S_{n(x)}^{m+1}(V(x)) \text{ for some } x \in Z \right\}$$

is a σ-discrete covering of Z. For $x \in Z$, let us denote

$$T(x) = \left(B_{i+1}(V(x)), B_{i+2}(V(x)), \dots, B_{\max\{i, n(x)\}}(V(x)) \right).$$

Now, whenever $y \in S^{m+1}_{n(x)}(V(x))$ for some $x \in Z$ and V' is a run satisfying $(\star^{T_0\hat{}T(x)})$ and giving y as its outcome then V' coincides with $V(x)$ in its first $n(x)$ moves, in particular $S^{m+1}_{n(x)}(V') = S^{m+1}_{n(x)}(V(x))$, and so $y \notin X \setminus \bigcup_{n=m+1}^{\infty} S^{m+1}_n(V')$ and $m+1$ is not a witness of Sisyfos' victory in the run V'. Thus, if $y \in S^{m+1}_{n(x)}(V(x)) \cap M^{T_0}_m$ then also $y \in M^{T_0\hat{}T(x)}_{m+1}$ which gives us the inclusion

$$S^{m+1}_{n(x)}(V(x)) \cap M^{T_0}_m \subseteq M^{T_0\hat{}T(x)}_{m+1}.$$

We can define

$$\mathcal{E} = \left\{M^{T_0}_{m+1}\right\} \cup \{E \cap M^{T_0}_m : E \in \mathcal{E}'\}$$

and

$$N^{T_0}_m = M^{T_0}_m \setminus \left(Z \cup M^{T_0}_{m+1}\right).$$

The system $\mathcal{E}$ is obviously σ-discrete and $M^{T_0}_m = N^{T_0}_m \cup \bigcup \mathcal{E}$. It only remains to show that the set $N^{T_0}_m$ is P-porous. Let us choose $x \in N^{T_0}_m$ arbitrarily. Then $x \in M^{T_0}_m \setminus M^{T_0}_{m+1}$ and so there exists a run V satisfying $(\star^{T_0})$ and giving x as its outcome such that $m+1$ is a witness of Sisyfos' victory in the run V, in particular

$$P\left(x, X \setminus \bigcup_{n=m+1}^{\infty} S^{m+1}_n(V)\right).$$

But

$$N^{T_0}_m \subseteq X \setminus Z \subseteq X \setminus \bigcup_{n=m+1}^{\infty} S^{m+1}_n(V),$$

and by (P1) we have $P\left(x, N^{T_0}_m\right)$. □

Theorem 3.3. *Sisyfos has a winning strategy in the game $G(A)$ if and only if A is a σ-P-porous set.*

Proof. First, let us assume that $A = \bigcup_{n=1}^{\infty} A_n$ where A_n is a P-porous set for $n \in \mathbb{N}$. In his nth move, let Sisyfos play $S^j_n = \emptyset$ for $j < n$ and $S^n_n = B_n \setminus \overline{A_n}$. Let Boulder and Sisyfos play a run of the game $G(A)$ such that Sisyfos keeps on the described strategy. Let $x \in X$ be an outcome of this run. We may assume that $x \in A$ because otherwise Sisyfos wins by condition (i) (see page 9). Then there exists $m \in \mathbb{N}$ such that $x \in A_m$. We have

$$X \setminus \bigcup_{n=m}^{\infty} S^m_n = \overline{A_m} \cup (X \setminus B_m).$$

Therefore

$$x \in A_m \subseteq X \setminus \bigcup_{n=m}^{\infty} S_n^m.$$

Further, P-porosity of A_m implies that $P(x, A_m)$. But this is equivalent to $P\left(x, \overline{A_m}\right)$ by (P3) and this is equivalent to $P\left(x, \overline{A_m} \cup (X \setminus B_m)\right)$ by (P2) since $x \in B_m$. So we also have $P\left(x, X \setminus \bigcup_{n=m}^{\infty} S_n^m\right)$. Therefore Sisyfos wins by condition (ii) (see page 9) with m as a witness and the described strategy is winning.

Now, let us assume that Sisyfos has a winning strategy σ in the game $G(A)$. Let us denote $E_0 = A$. By Lemma 3.2, we have

$$A = E_0 = M_0^{\emptyset} = N_0^{\emptyset} \cup \bigcup \mathcal{E}$$

where $N_0^{\emptyset}$ is P-porous and $\mathcal{E}$ is a σ-discrete system of sets such that for every $E_1 \in \mathcal{E}$, there exists a good sequence $T(E_1)$ such that $E_1 \subseteq M_1^{T(E_1)}$. Now, for every $E_1 \in \mathcal{E}$ we have

$$E_1 \subseteq M_1^{T(E_1)} = N_1^{T(E_1)} \cup \bigcup \mathcal{F}^{E_1}$$

where $N_1^{T(E_1)}$ is P-porous and $\mathcal{F}^{E_1}$ is a σ-discrete system of sets such that for every $E_2 \in \mathcal{F}^{E_1}$, there exists a finite sequence $T(E_1, E_2)$ of open balls such that $T(E_1) {}^\wedge T(E_1, E_2)$ is good and $E_2 \subseteq M_2^{T(E_1)^\wedge T(E_1,E_2)}$. If we denote

$$\mathcal{E}^{E_1} = \left\{E_1 \cap E_2 \colon E_2 \in \mathcal{F}^{E_1}\right\}$$

then we have

$$E_1 = \left(E_1 \cap N_1^{T(E_1)}\right) \cup \bigcup \mathcal{E}^{E_1}.$$

In the third step, for every $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}^{E_1}$ we have

$$E_2 \subseteq M_2^{T(E_1)^\wedge T(E_1,E_2)} = N_2^{T(E_1)^\wedge T(E_1,E_2)} \cup \bigcup \mathcal{F}^{E_1,E_2}$$

where $N_2^{T(E_1)^\wedge T(E_1,E_2)}$ is P-porous and $\mathcal{F}^{E_1,E_2}$ is a σ-discrete system of sets such that for every $E_3 \in \mathcal{F}^{E_1,E_2}$, there exists a finite sequence $T(E_1, E_2, E_3)$ of open balls such that $T(E_1) {}^\wedge T(E_1, E_2) {}^\wedge T(E_1, E_2, E_3)$ is good and $E_3 \subseteq M_3^{T(E_1)^\wedge T(E_1,E_2)^\wedge T(E_1,E_2,E_3)}$. If we denote

$$\mathcal{E}^{E_1,E_2} = \left\{E_2 \cap E_3 \colon E_3 \in \mathcal{F}^{E_1,E_2}\right\}$$

then we have

$$E_2 = \left(E_2 \cap N_2^{T(E_1)^\wedge T(E_1,E_2)}\right) \cup \bigcup \mathcal{E}^{E_1,E_2}.$$

By iterating this process, we get a system of sets

$$\mathcal{U} = \big\{E_k \cap N_k^{T(E_1)^\wedge T(E_1,E_2)^\wedge \dots ^\wedge T(E_1,E_2,\dots,E_k)}\colon$$
$$k \in \mathbb{N}\cup\{0\}, E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \dots, E_k \in \mathcal{E}^{E_1,E_2,\dots,E_{k-1}}\big\}$$

such that for every $k \in \mathbb{N}\cup\{0\}$ and for every $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \dots, E_k \in \mathcal{E}^{E_1,E_2,\dots,E_{k-1}}$, the sequence $T(E_1)^\wedge T(E_1,E_2)^\wedge \dots {}^\wedge T(E_1,E_2,\dots,E_k)$ is good and such that every $U \in \mathcal{U}$ is P-porous.

We show that $A \subseteq \bigcup \mathcal{U}$. Suppose that this is not true. Then there exist $x \in A$ and a sequence $(E_k)_{k=1}^{\infty}$, $E_k \in \mathcal{E}^{E_1,E_2,\dots,E_{k-1}}$, $k \in \mathbb{N}$, such that

$$x \in E_k \subseteq M_k^{T(E_1)^\wedge T(E_1,E_2)^\wedge \dots ^\wedge T(E_1,E_2,\dots,E_k)}$$

for every $k \in \mathbb{N}$. Therefore Boulder can play a run of the game $G(A)$ in the following way. He plays all the balls from $T(E_1)$ in sequence on his first moves, then all the balls from $T(E_1,E_2)$ and so on. (It can also happen that this process is finished after finitely many moves. This becomes in the case that there exists $k_0 \in \mathbb{N}\cup\{0\}$ such that the sequences $T(E_1,E_2,\dots,E_k)$ are empty for $k \geq k_0$. Then Boulder can finish the run arbitrarily such that the outcome of the run is x.) After such a run, x is its outcome and any $m \in \mathbb{N}$ is not a witness of Sisyfos' victory as long as Sisyfos keeps on the strategy σ since $x \in M_m^{T(E_1)^\wedge T(E_1,E_2)^\wedge \dots ^\wedge T(E_1,E_2,\dots,E_m)}$ for every $m \in \mathbb{N}$. This is a contradiction with the assumption that the strategy σ is winning for Sisyfos.

By (P1), it suffices to show that $\bigcup \mathcal{U}$ is a σ-P-porous set. We have

$$\bigcup \mathcal{U} = \bigcup_{k=0}^{\infty} \bigcup \mathcal{U}_k$$

where

$$\mathcal{U}_k = \big\{E_k \cap N_k^{T(E_1)^\wedge T(E_1,E_2)^\wedge \dots ^\wedge T(E_1,E_2,\dots,E_k)}\colon$$
$$E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \dots, E_k \in \mathcal{E}^{E_1,E_2,\dots,E_{k-1}}\big\}.$$

We will prove that $\bigcup \mathcal{U}_k$ is a σ-P-porous set for every $k \in \mathbb{N}\cup\{0\}$ which is obviously sufficient. For $k = 0$ we know that $\bigcup \mathcal{U}_0 = N_0^{\emptyset}$ which is a P-porous set. Suppose that $k > 0$ and $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \dots, E_{k-1} \in \mathcal{E}^{E_1,E_2,\dots,E_{k-2}}$ are fixed. Then

$$C(E_1,E_2,\dots,E_{k-1}) :=$$
$$\bigcup \left\{E_k \cap N_k^{T(E_1)^\wedge T(E_1,E_2)^\wedge \dots ^\wedge T(E_1,E_2,\dots,E_k)}\colon E_k \in \mathcal{E}^{E_1,E_2,\dots,E_{k-1}}\right\}$$

is a union of a σ-discrete system (since $\mathcal{E}^{E_1,E_2,\dots,E_{k-1}}$ is σ-discrete) of P-porous sets and by Lemma 3.1 it is a σ-P-porous set. Next, if only $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \dots,$ $E_{k-2} \in \mathcal{E}^{E_1,E_2,\dots,E_{k-3}}$ are fixed, then

$$C\left(E_1, E_2, \ldots, E_{k-2}\right) :=$$

$$\bigcup\left\{E_k \cap N_k^{T(E_1)^\wedge T(E_1,E_2)^\wedge \ldots ^\wedge T(E_1,E_2,\ldots,E_k)} : E_{k-1} \in \mathcal{E}^{E_1,E_2,\ldots,E_{k-2}}, E_k \in \mathcal{E}^{E_1,E_2,\ldots,E_{k-1}}\right\}$$

$$= \bigcup\left\{C\left(E_1, E_2, \ldots, E_{k-1}\right) : E_{k-1} \in \mathcal{E}^{E_1,E_2,\ldots,E_{k-2}}\right\}$$

is a union of a σ-discrete system (indeed, $C\left(E_1, E_2, \ldots, E_{k-1}\right) \subseteq E_{k-1}$ and $\mathcal{E}^{E_1,E_2,\ldots,E_{k-2}}$ is σ-discrete) of σ-P-porous sets and by Lemma 3.1 it is σ-P-porous again. Repeating this consideration long enough, we get that only for $E_1 \in \mathcal{E}$ fixed,

$$C\left(E_1\right) :=$$

$$\bigcup\left\{E_k \cap N_k^{T(E_1)^\wedge T(E_1,E_2)^\wedge \ldots ^\wedge T(E_1,E_2,\ldots,E_k)} : E_2 \in \mathcal{E}^{E_1}, E_3 \in \mathcal{E}^{E_1,E_2}, \ldots, E_k \in \mathcal{E}^{E_1,E_2,\ldots,E_{k-1}}\right\}$$

$$= \bigcup\left\{C\left(E_1, E_2\right) : E_2 \in \mathcal{E}^{E_1}\right\}$$

is σ-P-porous as a union of a σ-discrete system of σ-P-porous sets. Finally,

$$\bigcup \mathcal{U}_k = \bigcup \left\{C\left(E_1\right) : E_1 \in \mathcal{E}\right\}$$

is σ-P-porous, too. $\square$

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