Hereditary Properties of Graphs

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Hereditarnia of Peter Mihók
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First problem

Definition

The **detour order** (of a vertex \( v \)) of a graph \( G \) is the order of a longest path (beginning at \( v \)).

The **detour sequence of** \( G \) is a sequence consisting of the detour orders of its vertices.

A graph is called a **detour graph** if its detour sequence is constant.

Problem

*Find necessary and sufficient conditions for a given sequence of positive integers to be the detour sequence of some graph.*
Two well-known conjectures

\( \tau(G) \) - detour order of \( G \)

**PPC - Path Partition Conjecture**

For any graph \( G \) and arbitrary positive integers \( a, b \) satisfying \( a + b = \tau(G) \), the vertices of a graph \( G \) can be partitioned into two parts \( A \) and \( B \) in such a way that the order of a longest path in \( G[A] \) is at most \( a \) and the order of a longest path in \( G[B] \) is at most \( b \).

A subset \( K \) of \( V \) is called a \( P_n \)-kernel of \( G \) if \( \tau(G[K]) \leq n - 1 \) and every vertex \( v \in V \setminus K \) is adjacent to an end-vertex of a path of order \( n - 1 \) in \( G[K] \).

**PKC - Path Kernel Conjecture**

Every graph has a \( P_k \)-kernel for every positive integer \( k \geq 2 \).
Generalised colouring of graphs

Given a graph $G$ and properties $P_1, P_2, P_3, P_4$. We would like to assign colours to the vertices of $G$ according to this prescription.
Generalised colouring of graphs

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Definition

Let $\mathcal{I}^\omega$ and $\mathcal{I}$ denote the class of all simple countable graphs, simple finite graphs respectively. A graph property $\mathcal{P}$ is any isomorphism-closed nonempty subclass of $\mathcal{I}^\omega$ ($\mathcal{I}$).
Properties of graphs

Definition
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Definition
Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties. A vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colouring (partition) of a graph $G = (V, E)$ is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that each colour class $V_i$ induces a subgraph $G[V_i]$ having property $\mathcal{P}_i$.

If each of the $\mathcal{P}_i$’s, $i = 1, 2, \ldots, n$, is the property $\mathcal{O}$ of being edgeless, we have the well-known proper vertex $n$-colouring.
Reducible properties of graphs

Notation

Let us denote by $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n, \ n \geq 2$ the set of all $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable graphs.

The binary operation $\circ$ is obviously commutative, associative on the class of graph properties and $\mathcal{E} = \{K_0\}$ is its neutral element.

Definition

A nontrivial graph property $\mathcal{P}$ is said to be reducible if there exist nontrivial graph properties $\mathcal{P}_1, \mathcal{P}_2$, such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$; otherwise $\mathcal{P}$ is called irreducible.

Problem

What properties are reducible?
Reducible properties of graphs

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A nontrivial graph property \( \mathcal{P} \) is said to be reducible if there exist nontrivial graph properties \( \mathcal{P}_1, \mathcal{P}_2 \), such that \( \mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2 \); otherwise \( \mathcal{P} \) is called irreducible.

Problem

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Reducible properties of graphs

Notation

Let us denote by \( R = P_1 \circ P_2 \circ \cdots \circ P_n, \ n \geq 2 \) the set of all \((P_1, P_2, \ldots, P_n)\)-colourable graphs.

The binary operation \( \circ \) is obviously commutative, associative on the class of graph properties and \( E = \{K_0\} \) is its neutral element.

Definition

A nontrivial graph property \( P \) is said to be reducible if there exist nontrivial graph properties \( P_1, P_2 \), such that \( P = P_1 \circ P_2 \); otherwise \( P \) is called irreducible.

Problem

What properties are reducible?
Definition

Let $\leq$ be a partial order on the set $\mathcal{I}$. Then we say that a property $\mathcal{P} \subseteq \mathcal{I}$ is $\leq$-hereditary if from the facts $G \in \mathcal{P}$ and $H \leq G$ follows that $H \in \mathcal{P}$.

A property $\mathcal{P}$ is said to be induced-hereditary if it is closed under taking induced subgraphs.

A property $\mathcal{P}$ is said to be hereditary if it is closed under taking subgraphs.
**Definition**

Let $\mathcal{P}$ be a graph property, $\mathcal{P}$ is of finite character if a graph in $\mathcal{I}^\omega$ has property $\mathcal{P}$ if and only if each its finite induced subgraph has property $\mathcal{P}$.

It is easy to see that if $\mathcal{P}$ is of finite character and a graph has property $\mathcal{P}$ then so does every induced subgraph.
Special properties of graphs

Definition

Let $\mathcal{P}$ be a graph property, $\mathcal{P}$ is of finite character if a graph in $\mathcal{I}^\omega$ has property $\mathcal{P}$ if and only if each its finite induced subgraph has property $\mathcal{P}$.

Thus properties of finite character are induced-hereditary. However not all induced-hereditary properties are of finite character.
Induced-hereditary properties of graphs

Example

The properties
- to be edgeless,
- to be of maximum degree at most $k$,
- to be $K_n$-free,
- to be acyclic,
- to be complete,
- to be perfect

are properties of finite character.
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Additivity

**Definition**

A property of graphs is called **additive** if it is closed with respect to taking disjoint union.

**Example**

The property **to be acyclic** is additive however the property **to be complete** it is not.
A property of graphs is called **additive** if it is closed with respect to taking disjoint union.

The property to be acyclic is additive however the property to be complete it is not.
Theorem (Cowen, Hechler, Mihók, 2002)

Let $G$ be a graph in $I^\omega$ and let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs of finite character. Then $G$ is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable if and only if every finite induced subgraph of $G$ is $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colourable.

It provides a generalisation of the classical result of de Bruijn and Erdős who proved that an infinite graph $G$ is $k$-colourable if and only if every finite subgraph of $G$ is $k$-colourable.
Theorem (Borowiecki and Mihók, 1991)

Partially ordered sets $L \subseteq = (L, \subseteq)$ and $L^a \subseteq = (L^a, \subseteq)$ form complete, distributive lattice with the least element $E$ and the greatest element $I$.

Analogous results can be proved for the sets $L \leq = (L, \leq)$ and $L^a \leq = (L^a, \leq)$. 
If $\alpha$ is any cardinal, then $L(\alpha)$ denotes the set of all hereditary properties that contains graphs with vertex set of order at most $\alpha$.

**Theorem (Jakubík, 2002)**

Partially ordered sets $L^a(\alpha)_{\subseteq} = (L^a(\alpha), \subseteq)$ and $L^a(\alpha)_{\leq} = (L(\alpha)^a, \leq)$ form completely distributive lattice with the least element $E$ and the greatest element $I$. $I$. $I$. $I$.
Three sisters
A characterisation in terms of forbidden substructures

Let $\mathcal{P} \subseteq \mathcal{I}$. We say that $\mathcal{P}$ can be characterized in terms of forbidden subgraphs if there exists a subset $\mathbf{F}(\mathcal{P})$ of $\mathcal{I}$ such that $G \in \mathcal{P}$ if and only if no subgraph of $G$ is in $\mathbf{F}(\mathcal{P})$.

Theorem (Greenwell, Hemminger and Klerlein, 1973)

Let $\mathcal{I}$ be a partially ordered class of graphs under $\subseteq$. A property $\mathcal{P}$ can be characterized in terms of forbidden subgraphs if and only if for all $G \in \mathcal{P}$, $H \subseteq G$ implies $H \in \mathcal{P}$. 
A characterisation in terms of forbidden substructures

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How to define hereditary properties

Minimal forbidden graphs

It is easy to see that if $F, F' \in F(\mathcal{P})$, $F \subseteq F'$ and we put $F'(\mathcal{P}) = F(\mathcal{P}) \setminus \{F'\}$, then $\mathcal{P}$ is uniquely determined by $F'(\mathcal{P})$, too.

**Definition**

The set of **minimal forbidden subgraphs** of $\mathcal{P}$ is the set:

$$F(\mathcal{P}) = \{G \in \mathcal{I} \setminus \mathcal{P} : \text{each proper subgraph of } G \text{ belongs to } \mathcal{P}\}.$$

Note that $F(\mathcal{P})$ may be finite or infinite. A property is additive if and only if $F(\mathcal{P})$ contains only connected graphs.
How to define hereditary properties

Minimal forbidden graphs

\( P \)
How to define hereditary properties

Minimal forbidden graphs

F(P)

P
How to define hereditary properties

Minimal forbidden graphs

$\mathcal{F}(\mathcal{P})$
Minimal forbidden graphs
How to define hereditary properties

Examples of minimal forbidden graphs

Example

Let $S_n$ and $P_n$ denote the star and the path on $n$ vertices, respectively. Then

- $F(\emptyset) = \{K_2\}$;
- $F(\emptyset_k) = \{H \in I : H \text{ is a tree on } k + 2 \text{ vertices}\}$,
- $F(S_k) = \{S_{k+2}\}$;
- $F(W_k) = \{P_{k+2}\}$;
- $F(T_k) = \{H \in I : H \text{ is homeomorphic to } K_{k+2} \text{ or } K_{\lceil \frac{k+3}{2} \rceil, \lfloor \frac{k+3}{2} \rfloor}\}$;
- $F(I_k) = \{K_{k+2}\}$.
For a non-negative integer $k$ and a graph $G$, we denote the set of all vertices of $G$ of degree $k+1$ by $M(G)$. If $S \subseteq V(G)$ is a cutset of vertices of $G$ and $G_1, G_2, \ldots, G_s, s \geq 2$ are the components of $G - S$, then the graph $G - V(G_i)$ is denoted by $H_i, i = 1, 2, \ldots, s$.

**Theorem (Mihók, 1981)**

A graph $G$ belongs to $\mathbf{F}(\mathcal{D}_k)$ if and only if

1. $G$ is connected,
2. $\delta(G) \geq k + 1$,
3. $V(G) \setminus M(G)$ is an independent set of vertices of $G$,
4. and for each cutset $S \subseteq V(G) \setminus M(G)$ we have that $\delta(H_i) \leq k$ for each $i = 1, 2, \ldots, s$. 
An extension of Brook’s theorem

**Theorem (Mihók, 1992)**

Let $T$ be a tree on $k + 2$ vertices, $k \geq 3$. If a connected graph $G$ does not contain $T$, then $G$ is $k$-colorable unless $G = K_{k+1}$. 
Maximal graphs

Any monotone property is uniquely determined by its set of minimal forbidden subgraphs. An alternative way is to characterize $\mathcal{P}$ by the set of graphs containing all the graphs in $\mathcal{P}$ as subgraphs.

Definition

To be more accurate, let us define the set of $\mathcal{P}$-maximal graphs by

$$M(\mathcal{P}) = \{ G \in \mathcal{P} : G + e \notin \mathcal{P} \text{ for each } e \in E(G) \}$$

and the set of $\mathcal{P}$-maximal graphs of order $n$ by

$$M(\mathcal{P}, n) = \{ G \in \mathcal{P} : |V(G)| = n \text{ and } G \in M(\mathcal{P}) \}.$$ 

From these definitions it is evident that $M(\mathcal{P}) = \bigcup_{n \geq 1} M(\mathcal{P}, n)$. 
Maximal graphs
Maximal graphs

\[ \mathcal{F}(\mathcal{P}) \]

\[ \mathcal{M}(\mathcal{P}) \]
Maximal graphs

\[ F(P) \]
\[ M(P) \]
\[ \bullet G \]
\[ \mathcal{P} \]
Maximal graphs

\[ G + e \]

\[ \mathcal{P} \]

\[ \mathcal{F}(\mathcal{P}) \]

\[ \mathcal{M}(\mathcal{P}) \]
How to define hereditary properties

Examples of maximal graphs

Example

- \( M(O^2) = \{ G \in \mathcal{I} : G \text{ is complete bipartite} \} \);
- \( M(S_k) = \{ G \in \mathcal{I} : G \text{ is almost } k + 1\text{-regular} \} \);
- \( M(D_1) = \{ G \in \mathcal{I} : G \text{ is a tree} \} \);
- \( M(T_3) = \{ G \in \mathcal{I} : G \text{ is a triangulation of the plane} \} \);
- \( M(\mathcal{PT}_k) = \{ G \in \mathcal{I} : G \text{ is a partial } k\text{-tree} \} \);
- \( M(D_k) = \{ G \in \mathcal{I} : G \text{ is maximal } k\text{-degenerate} \} \);
- \( M(O_k) = \{ G \in \mathcal{I} : G = K_{i_1} \cup K_{i_2} \cup \cdots \cup K_{i_j} \text{ s.t. for each } r, s = 1, 2, \ldots, j, r \neq s \text{ holds } i_r \leq k + 1 \text{ and } i_r + i_s \geq k + 2 \} \).
Some relationships

Lemma (Borowiecki and Mihók, 1991)

Let $P_1, P_2$ be any hereditary properties. Then the following statements are mutually equivalent:

1. $P_1 \subseteq P_2$
2. for each $H \in F(P_2)$ there exists $H' \in F(P_1)$ such that $H' \subseteq H$;
3. for any positive integer $n$ and an arbitrary $G \in M(n, P_1)$ there is $G' \in M(n, P_2)$ such that $G \subseteq G'$. 
Generating sets

It is natural that the characterization of hereditary property in terms of $\mathcal{P}$-maximal graphs leads to a more general concept — the generating set of a hereditary (an additive hereditary) property.

Consider an arbitrary set $\mathcal{G}$, a subset of $\mathcal{I}$. It is quite easy to see that the property

$$[\mathcal{G}] = \{ G \in \mathcal{I} : G \text{ is a subgraph of some graph } H \in \mathcal{G} \}$$

is hereditary, and the property

$$[\mathcal{G}]^a = \{ G \in \mathcal{I} : \text{ each component of } G \text{ is a subgraph of some } H \in \mathcal{G} \}$$

is in addition additive and both are generated by the set $\mathcal{G}$, called generating set.
How to define hereditary properties

Generators

Definition

A graph $G$ (finite or infinite) is called strong generator of a property $P$ if $[\{G\}] = P$. 
How to define hereditary properties

Chains of hereditary properties

Definition

For a normalised invariant \( \varphi \) we define properties

\[ P_{(\varphi,k)} = \{ G \in \mathcal{I} : \varphi(G) \leq k \}, \quad k = 0, 1, \ldots \]

Then the chain

\[ P_{(\varphi,0)} \subset P_{(\varphi,1)} \subset \cdots \subset P_{(\varphi,n)} \subset \cdots \]

of hereditary properties is called the chain associated with (a normalised graph invariant) \( \varphi \).
Suppose that in the lattice $\mathbb{L}_\subseteq$ we have a finite or countable chain

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_k \subset \cdots, \quad k \geq 0.$$ 

Let

$$\mathcal{P} = \bigcup_{k \geq 0} \mathcal{P}_k.$$ 

Then an invariant $f: \mathcal{P} \to \mathbb{N}$ by this chain is defined as follows:

- $f(G) = 0$, if $G \in \mathcal{P}_0$,
- $f(G) = k$, if $G \in \mathcal{P}_k \setminus \mathcal{P}_{k-1}$, for $k \geq 1$. 
Problem

Given a family $F(P)$ of forbidden subgraphs, find the number

$$ex(n, P) = \max\{|E(G)| : G \in M(n, P)\}.$$ 

Definition

The $P$-maximal graphs of order $n$ with exactly $ex(n, P)$ edges are called $P$-extremal graphs.
A generalisation of Turan’s theorem

Theorem (Erdős and Stone, 1966)

If $\mathcal{P}$ is a hereditary property with chromatic number $\chi(\mathcal{P}) = \min\{\chi(F) : F \in \mathbf{F}(\mathcal{P})\}$, then

$$\text{ex}(n, \mathcal{P}) = \left(1 - \frac{1}{\chi(\mathcal{P}) - 1}\right) \binom{n}{2} + o(n^2).$$

The meaning of this theorem is that $\text{ex}(n, \mathcal{P})$ depends only very loosely on the structure of $\mathcal{P}$, it is already determined by $\chi(\mathcal{P})$. By the above theorem $\text{ex}(n, \mathcal{P}) = o(n^2)$ if and only if $\mathcal{P}$ is a degenerate property, i.e. its chromatic number is 2.
Saturated graphs

It is natural to investigate also the “opposite side”.

**Definition**

\[
\text{sat}(n, \mathcal{P}) = \min\{|E(G)| : G \in \mathcal{M}(n, \mathcal{P})\}.
\]

The graphs of order \(n\) with \(\text{sat}(n, \mathcal{P})\) edges are called \(\mathcal{P}\)-saturated.
Saturated graphs

Theorem (Kászonyi and Tuza, 1986)

If $\mathcal{P}$ is a given hereditary property and

1. $u = u(\mathcal{P}) = \min \left\{ |V(F)| - \beta(F) - 1 : F \in \mathbf{F}(\mathcal{P}) \right\}$,

2. $d = d(\mathcal{P}) = \min \left\{ |E(F')| : F' \subseteq F \in \mathbf{F}(\mathcal{P}) \text{ is induced by a set } S \cup \{x\}, S \subseteq V(F) \text{ is independent, } x \in V(F) \setminus S \text{ and } \right.$

$\left. |S| = |V(F)| - u - 1 \right\}$,

then

$$\text{sat}(n, \mathcal{P}) \leq un + \frac{1}{2} (d - 1)(n - u) - \binom{u + 1}{2},$$

if $n$ is large enough.
An intriguing problem

Problem

For what hereditary properties is true that:

\[ \text{ext}(n, \mathcal{P}) = \text{sat}(n, \mathcal{P})? \]

Some positive answers:

- planar graphs
- outerplanar graphs
- \( k \)-degenerate graphs.
Minimal reducible bounds - motivation
How to compare different graph colouring results

Minimal reducible bounds - motivation

$O^6

O^5$

$T_3$
Minimal reducible bounds - motivation
How to compare different graph colouring results

Minimal reducible bounds - motivation

\[ T_3 \]

\[ O^6 \]

\[ O^5 \]

\[ (O^2 \cap T_3)^2 \]

\[ O \circ (O^3 \cap T_3) \]
How to compare different graph colouring results

Minimal reducible bounds - motivation

\[ \mathcal{O}^6 \]
\[ \mathcal{O}^5 \]
\[ \mathcal{O}^4 \]
\[ (\mathcal{O}^2 \cap \mathcal{T}_3)^2 \]
\[ (\mathcal{O} \cap \mathcal{T}_3)^2 \]
\[ \mathcal{O} \circ (\mathcal{O}^3 \cap \mathcal{T}_3) \]

\[ (\mathcal{S}_2 \cap \mathcal{D}_1)^3 \]

\[ \mathcal{D}_1 \circ \mathcal{D}_2 \]
How to compare different graph colouring results

Minimal reducible bounds - motivation

\[(S_2 \cap D_1)^3\]
\[(O^2 \cap T_3)^2\]
\[O^6\]
\[O^5\]
\[O^4\]
\[O^\circ (O^3 \cap T_3)\]
\[D_1^\circ D_2\]
How to compare different graph colouring results

Minimal reducible bound

Definition

A reducible property $R \in L^a \subseteq (\subseteq L^a)$ is called a minimal reducible bound for a property $P \in L^a \subseteq (\subseteq L^a)$ if $P \subseteq R$ and there is no reducible property $R^* \in L^a \subseteq (\subseteq L^a)$ satisfying $P \subseteq R^* \subset R$.

This means that the obtained colouring result is sharp and in some sense cannot be improved.

The problem of finding all minimal reducible bounds for the class of planar graphs was formulated by Mihók and Toft in 1993 (Problem 17.9)
Theorem (Berger, 2001)

- Every additive hereditary property has at least one minimal reducible bound.
- All the reducible bounds of a property $P$ contain a minimal reducible bound for $P$.
- Every reducible additive hereditary property is a minimal reducible bound for some irreducible additive hereditary property.

Theorem

Let $P$ be an additive induced-hereditary (hereditary) property of graphs with $\chi(P) = k$. Then all the minimal reducible bounds for $P$ are properties consisting of at most $k$ irreducible factors.
Theorem (Berger, 2001)

- Every additive hereditary property has at least one minimal reducible bound.
- All the reducible bounds of a property \( P \) contain a minimal reducible bound for \( P \).
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Theorem

Let \( P \) be an additive induced-hereditary (hereditary) property of graphs with \( \chi(P) = k \). Then all the minimal reducible bounds for \( P \) are properties consisting of at most \( k \) irreducible factors.
Two interesting results

\[ \mathcal{O} \circ \mathcal{D}_1 \quad (\mathcal{D}_1 \cap S_2)^2 \]

\[ \mathcal{T}_2 \]
Two interesting results

\[ O \circ D_1 \quad (D_1 \cap S_2)^2 \quad D_1 \circ D_{k-2} \quad \ldots \quad D_{\left\lfloor \frac{k-1}{2} \right\rfloor} \circ D_{\left\lceil \frac{k-1}{2} \right\rceil} \]
Some results

**Theorem**

- $B_L \subseteq (O_k) = \{O_p \circ O_q : p + q + 1 = k\}$;
- $B_L \subseteq (D_k) = \{D_p \circ D_q : p + q + 1 = k\}$;
- $B_L \subseteq (PT_k) = \{PT_p \circ PT_q : p + q + 1 = k\}$;
- $B_L \subseteq (T_2) = \{O \circ D_1, (D_1 \cap S_2)^2\}$;
- $B_L \subseteq (D_k) = \{D_p \circ D_q : p + q + 1 = k\}$. 
How to compare different graph colouring results

A general criteria

Theorem

Let $\mathcal{O} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_k$, $k \geq 1$ be a chain of additive induced-hereditary degenerate properties of graphs. If for arbitrary non-negative integers $r, s, t, u$, $r + s + 1 = k$, $t + u = k$ the properties $\mathcal{P}_r, \mathcal{P}_s, \mathcal{P}_t, \mathcal{P}_u$ satisfy the following two conditions

(i) $\mathcal{P}_k \subseteq \mathcal{P}_r \circ \mathcal{P}_s$;

(ii) $\mathcal{P}_k \overset{i}{\rightarrow} (\mathcal{P}_t, \mathcal{P}_u)$,

then the set of minimal reducible bounds for $\mathcal{P}_k$ in the lattice $L^a \leq$ is of the form $B_{L^a} (\mathcal{P}_k) = \{ \mathcal{P}_p \circ \mathcal{P}_q : p + q + 1 = k \}$. 

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Hereditary Properties of Graphs

Hereditarnia of Peter Mihók
A characterisation of hereditary properties is difficult

- A. Berger proved that any reducible additive hereditary property of graphs has infinitely many minimal forbidden graphs.
- Only very little is known about the structure of $F(P \circ Q)$, even in the case when the structure of $F(P)$ and $F(Q)$ is known.

Useful information on the structure of $F(P \circ Q)$ can be obtained by investigation of graph invariants associated with a property $P \circ Q$. 
By a **graph invariant** \( \varphi \) we mean any integer-valued (real-valued) function defined on \( \mathcal{I} \) such that \( \varphi(G) = \varphi(H) \) for each pair \( G, H \) of isomorphic graphs.

**Definition**

We say that the invariant \( \varphi \) interpolates over the class \( \mathcal{P} \) of graphs if for any \( G, H \in \mathcal{P} \) and each integer \( k \) between \( \varphi(G) \) and \( \varphi(H) \) there exists a graph \( F \in \mathcal{P} \) such that \( \varphi(F) = k \). A graph invariant \( \varphi \) is called **monotone** if \( G \subseteq H \) implies \( \varphi(G) \leq \varphi(H) \). A graph invariant \( \varphi \) is called **additive** if \( \varphi(G \cup H) \leq \max\{\varphi(G), \varphi(H)\} \) for any graphs \( G, H \).
By a **graph invariant** $\varphi$ we mean any integer-valued (real-valued) function defined on $\mathcal{I}$ such that $\varphi(G) = \varphi(H)$ for each pair $G, H$ of isomorphic graphs.

**Definition**

We say that the invariant $\varphi$ **interpolates over the class** $\mathcal{P}$ **of graphs** if for any $G, H \in \mathcal{P}$ and each integer $k$ between $\varphi(G)$ and $\varphi(H)$ there exists a graph $F \in \mathcal{P}$ such that $\varphi(F) = k$. A graph invariant $\varphi$ is called **monotone** if $G \subseteq H$ implies $\varphi(G) \leq \varphi(H)$. A graph invariant $\varphi$ is called **additive** if $\varphi(G \cup H) \leq \max\{\varphi(G), \varphi(H)\}$ for any graphs $G, H$. 
Given a graph invariant $\varphi$, we define the associated invariant of a property $P$ in the following manner:

$$
\varphi(P) = \min \{ \varphi(F) : F \in F(P) \}.
$$

**Definition**

We say that a graph invariant $\varphi$ is additive with respect to reducible hereditary properties (abbreviated by ARHP) if for any reducible property $P \circ Q$ the equality $\varphi(P \circ Q) = \varphi(P) + \varphi(Q)$ is valid.
Given a graph invariant \( \varphi \), we define the associated invariant of a property \( \mathcal{P} \) in the following manner:

\[
\varphi(\mathcal{P}) = \min\{\varphi(F) : F \in \mathcal{F}(\mathcal{P})\}.
\]

**Definition**

We say that a graph invariant \( \varphi \) is **additive with respect to reducible hereditary properties** (abbreviated by **ARHP**) if for any reducible property \( \mathcal{P} \circ \mathcal{Q} \) the equality \( \varphi(\mathcal{P} \circ \mathcal{Q}) = \varphi(\mathcal{P}) + \varphi(\mathcal{Q}) \) is valid.
A graph invariant $\varphi(\mathcal{P})$ strongly depends on the features of the minimal forbidden subgraphs.

**Lemma**

Let $\varphi$ be a monotone graph invariant such that, for every graph $G$ and every edge $e$ from its complement, $\varphi(G + e) \leq \varphi(G) + 1$ and let $\mathcal{P}$ be a hereditary graph property. Then for any graph $G \in \mathcal{M}(n, \mathcal{P})$ with $n \geq c(\mathcal{P}) + 2$ the following holds: $\varphi(G) \geq \varphi(\mathcal{P}) - 1$. 
For a graph invariant $\varphi$ we can define an associated graph invariant $\hat{\varphi}$ in the following way:

$$\hat{\varphi}(G) = \max_{H \subseteq G} \varphi(H).$$

**Lemma**

*If for some invariant $\varphi$ there is an constant $c$, such that the equality $\varphi(Q_1 \circ Q_2) = \varphi(Q_1) + \varphi(Q_2) + c$ holds for all hereditary properties $Q_1$ and $Q_2$, then the invariant $\varphi^*(G) = \varphi(G) + c$ is ARHP.*
A characterisation of ARHP invariants - cont.

**Theorem**

Let \( \varphi \) be a normalised graph invariant and \( \mathcal{P}(\varphi,0) \subset \mathcal{P}(\varphi,1) \subset \ldots \) be the chain associated with \( \varphi \). Then \( \varphi \) is additive with respect to hereditary properties if and only if for every pair of non-negative integers \( k, l \) the following conditions hold:

(i) \( \mathcal{P}(\varphi,k+l+1) \subseteq \mathcal{P}(\varphi,k) \circ \mathcal{P}(\varphi,l) \);

(ii) \( \mathcal{P}(\varphi,k+l) \to (\mathcal{P}(\varphi,k), \mathcal{P}(\varphi,l)) \).
Examples of ARHP invariants

Example

The following invariants are ARHP

- $p^*(G) = p(G) - 1$, where $p(G)$ denotes the order of a graph $G$;
- $\pi^*(G)$, where $\pi(G)$ is the size of the largest tree contained in a graph $G$;
- the subchromatic number $\psi(G) = \chi(G) - 1$;
- the degeneracy number $\hat{\delta}(G) = col(G) - 1$ (sometimes denoted also by $\rho(G)$);
- the tree-width $tw$
Properties and invariants

Conjecture on sub-choice number

It is well known, that for any graph $G$ we have

$$\chi(G) \leq ch(G) \leq col(G) \leq \Delta(G) + 1.$$  

We proved that the subchromatic number $\psi$ and the degeneracy number $\hat{\delta}(G)$ are ARHP. We conjecture that the ‘sub-choice number’ $ch^*(G) = ch(G) - 1$ is ARHP, too.

Conjecture

The sub-choice number $ch^*(P) = ch(G) - 1$ is ARHP.
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**Conjecture**

The sub-choice number $ch^*(\mathcal{P}) = ch(G) - 1$ is ARHP.
Unique Factorisation Problem

Problem

Given a reducible additive property of finite character $\mathcal{R}$. Does there exist a factorisation of $\mathcal{R}$ into finite number of irreducible factors? Is the factorisation unique?

This problem states a natural question that provides an analogue of Fundamental Theorem of Arithmetics.
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History of UFT

- 1993 - Mihók, G.S. - UFT for properties without $K_4$
- 1995 - Mihók, Vasky - UFT for properties without $K_5$
- 1996 - Kratochvíl, P.M. - UFT for hom-properties
- 1997 - Mihók, G.S., Vasky - UFT for additive hereditary properties
- 2000 - Mihók - UFT for additive induced-hereditary properties
- 2003 - Farrugia, Mihók, Richter, G.S. - UFT from compositive induced-hereditary properties
- 2003 - Imrich, Mihók, G.S. - UFT for properties of finite character
The main problem of the proof of UFT is the construction of irreducible factors. This process is rather technical and complicated and the proof of the uniqueness requires another non-trivial effort.

**Observation**

FCA allows us to identify and describe the irreducible factors in very natural way and therefore it can help to understand the proof and provide a suggestion how to find the factors algorithmically.
What is the benefit from an application of FCA?

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**Observation**

FCA allows us to identify and describe the irreducible factors in very natural way and therefore it can help to understand the proof and provide a suggestion how to find the factors algorithmically.
Context for graph properties

Definition

Let us define a context \((O, M, I)\) by setting objects to countable simple graphs, e.g. \(O = \mathcal{I}^\omega\). For each connected finite simple graph \(F \in \mathcal{I}\) let us consider an attribute \(m_F : \text{do not contain an induced-subgraph isomorphic to } F\).

Thus \(Glm_F\) means that the graph \(G\) does not contain any induced subgraph isomorphic to \(F\).

For an object \(g \in O\) we write \(g' = \{ m \in M | glm \}\) and \(\gamma g\) for the object concept \((g'', g')\), where \(g'' = \{\{g\}'\}'\).
Unique Factorisation Theorem

An illustration

Example

- If $G$ is planar then $Glm_{K_5}$.
- If $G$ is acyclic then $Glm_{C_n}$, for any positive $n \geq 3$, where $C_n$ denotes the cycle of order $n$.
- For any tree $T$ of order $n$, it is not valid $K_nlm_T$ because $K_n$ contains all trees of order at most $n$ as a subgraph.
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Example

- If $G$ is planar then $Glm_{K_5}$.
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- For any tree $T$ of order $n$, it is not valid $K_nIm_T$ because $K_n$ contains all trees of order at most $n$ as a subgraph.
Theorem

Every reducible additive property $R$ of finite character is uniquely factorisable into finite number of irreducible factors belonging to $\mathbb{M}^\omega$.

Remark

The assumption of additivity is substantial.
Uniquely partitionable graphs

**Theorem**

Let $\mathcal{P}$ be a hereditary property and $n$ be a positive integer. Then a uniquely $\mathcal{P}^n$-partitionable graph exists if and only if $\mathcal{P}$ is irreducible.
A generalisation of UFT

One can verify that for the used method of the proof it is not important that we are dealing with simple graphs. Therefore, the proof can be applied for directed graphs, hypergraphs or partially ordered sets, i.e. relational structures.

Definition

A concrete category $\mathbf{C}$ is a collection of objects and arrows called morphisms. An object in a concrete category $\mathbf{C}$ is a set with structure. The morphism between two objects is considered to be a structure preserving mapping.

The natural examples of concrete categories are: Set of sets, FinSet of finite sets, Graph of graphs, Poset of partially ordered sets with structure preserving mappings, i.e. the homomorphisms of corresponding structures.
Unique Factorisation Theorem

**UFT for system of objects**

**Theorem**

*Every reducible additive property \( \mathcal{R} \) of object-systems of finite character is uniquely factorisable into finite number of irreducible factors.*
Thank you very much for your attention