On heterogeneous formal contexts
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Abstract
We propose a new type of fuzzification for formal concept analysis that works with heterogeneous values in a context and illustrate this with an example. We formulate and prove an appropriate counterpart to the so-called basic theorem of a concept lattice. We show that this is a generalization of the previous approaches: it covers the so-called generalized concept lattice and multi-adjoint t-concept lattices.

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1. Introduction

Formal concept analysis (FCA) is a data-mining method applied to a rectangular matrix of data in which each row corresponds to some object, each column corresponds to some possible attribute, and the matrix field value denotes a membership of the column attribute for row object. One of the goals of this method is to find so-called concepts, which are stable (in some sense) pairs of subsets of objects and attributes. The method can be considered a nice application of the algebraic notion of a Galois connection. It has been described in detail by Ganter and Wille [12], in particular for the so-called crisp case with binary matrix data. A natural question that arises is what happens if the matrix data are non-binary.

Besides the conceptual scaling that returns concepts with crisp subsets in both coordinates [12], some other approaches return concepts with fuzzy subsets in at least one coordinate. The first was provided by Burusco and Fuentes-Gonzalez [9] and was independently improved by Bělohlávek [2,3] and Pollandt [25,26] using values from the same residual lattice for matrix values and for the fuzziness of subsets of the objects and attributes. Another approach independently proposed by Ben Yahia and Jaoua [8], Bělohlávek et al. [5] and Krajčí [13] is not as symmetric; it considers fuzzy subsets in one coordinate and crisp (binary) subsets in another one. All these approaches are covered by a common platform, the so-called generalized concept lattice [15,16], which diversifies the fuzziness of subsets of the attributes, the fuzziness of subsets of the objects, and the fuzziness of the matrix values.

Medina and Ojeda-Aciego then applied the idea of multi-adjointness used in logic-programming [20–22] to FCA [17,19]. Because of its novelty and originality, this approach is not (at least immediately) covered by generalized concept lattices. This inspired us to modify our previous approach so that it will work with different mutual relationships between the objects and attributes. We also use different lattices for the different rows and columns and the matrix data.

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The usefulness of this approach in practice is demonstrated by an example of personal accommodation requirements. In comparison with previous methods that work for attributes and objects of the same type, an important advantage of our new approach is the possibility to apply FCA to heterogeneous data. Thus, we call this new approach heterogeneous.

2. Heterogeneous formal context

Consider the following situation. People who are going to stay at a cottage together have different preferences for conditions, depending on the number of days spent at the cottage. One may prefer hot water, whereas others may expect an internet connection. The following requirements are examples:

- Eva will accept large discomfort regarding hot water availability and partial discomfort for an internet/TV connection.
- John will accept partial discomfort regarding hot water availability and large discomfort for an internet/TV connection.
- Eva will not accept the absence of hot water on one arbitrary day at the cottage.
- John will not accept the absence of hot water only on his second day.

The problem is to identify which conditions have to be fulfilled to satisfy all the people staying at the cottage for different numbers of days.

We cannot cover this type of complex information using existing approaches. Therefore, we define a heterogeneous formal context to formally describe the above situation.

Let $A$ and $B$ be non-empty sets. Let $\mathcal{P} = \{(P_a,b, \leq_{P_a,b}) : a \in A, b \in B\}$ be a system of posets and let $R$ be a function from $A \times B$ such that $R(a, b) \in P_a,b$ for all $a \in A$ and $b \in B$. Let $\mathcal{C} = \{(C_a, \leq_{C_a}) : a \in A\}$ and $\mathcal{D} = \{(D_b, \leq_{D_b}) : b \in B\}$ be systems of complete lattices. (For simplicity, we omit the indices for all $\leq$, since it is always clear which one is used.)

Let $\otimes = (\bullet_{a,b} : a \in A, b \in B)$ be a system of operations such that $\bullet_{a,b}$ is from $C_a \times D_b$ to $P_{a,b}$ and is isotone and left-continuous in both arguments, that is:

1. $c_1 \leq c_2$ implies $c_1 \bullet_{a,b} d \leq c_2 \bullet_{a,b} d$ for all $c_1, c_2 \in C_a$ and $d \in D_b$,
2. $d_1 \leq d_2$ implies $c \bullet_{a,b} d_1 \leq c \bullet_{a,b} d_2$ for all $c \in C_a$ and $d_1, d_2 \in D_b$,
3. If $c \bullet_{a,b} d \leq p$ for some $d \in D_b$, $p \in P_{a,b}$ and all $c \in X \leq C_a$, then sup $X \bullet_{a,b} d \leq p$,
4. If $c \bullet_{a,b} d \leq p$ for some $c \in C_a$, $p \in P_{a,b}$ and all $d \in Y \leq D_b$, then $c \bullet_{a,b} \sup Y \leq p$.

Then we call the tuple $(A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \otimes)$ a heterogeneous formal context.

Note that if $C_a = D_b$ and $\bullet_{a,b}$ is commutative, then these conditions can be reduced to the following two:

1. $c_1 \leq c_2$ implies $c_1 \bullet_{a,b} d \leq c_2 \bullet_{a,b} d$ for all $c_1, c_2, d \in C_a = D_b$, and
2. If $c \bullet_{a,b} d \leq p$ for some $d \in C_a$, $p \in P_{a,b}$ and all $c \in X \leq C_a = D_b$, then sup $X \bullet_{a,b} d \leq p$.

To illustrate such a heterogeneous formal context, let $B = \{\text{Eva}, \text{John}, \ldots\}$ be the set of objects consisting of people thinking about staying at a cottage. Let $A = \{\text{water}, \text{services}, \text{lake}\}$ be the set of attributes in relation to hot water, services and lake availability at the cottage. For simplicity, we consider just two people and two conditions from Fig. 1, but this heterogeneous method also works for arbitrary numbers of people and conditions. Eva has three preferences for staying: not at all, one day (it does not matter which one) or both days ($D_{\text{Eva}}$). John has four preferences: not at all, only Saturday, only Sunday, or both days ($D_{\text{John}}$). For the attributes, the water can be hot or cold ($C_{\text{water}}$) and there are four possibilities for services: internet and television, internet only, television only, or nothing at all ($C_{\text{services}}$). Each person can accept different degrees of discomfort for their preferences. Thus, the behavior of each person can be expressed as

$\otimes = (\bullet_{\text{water}, \text{Eva}}, \bullet_{\text{services}, \text{Eva}}, \bullet_{\text{water, John}}, \bullet_{\text{services, John}})$. Similarly $\mathcal{P} = (P_{\text{water, Eva}}, P_{\text{services, Eva}}, P_{\text{water, John}}, P_{\text{services, John}})$ expresses the degrees of discomfort for each person and each condition. For instance, $\bullet_{\text{services, Eva}}$ is from $C_{\text{services}} \times D_{\text{Eva}}$ to $P_{\text{services, Eva}}$, where $P_{\text{services, Eva}} = [0, 1/2, 1]$ denotes comfort, partial discomfort and large discomfort, respectively, for Eva.

The behavior of the isotone and left-continuous operations $\otimes = (\bullet_{\text{water, Eva}}, \bullet_{\text{services, Eva}}, \bullet_{\text{water, John}}, \bullet_{\text{services, John}})$ with respect to the number of days and cottage conditions is shown in Fig. 2.

We provide the following remarks and interpretation:

- Note that $c \bullet_{\text{services, Eva}} \emptyset = 0$ for all $c \in C_{\text{services}}$, because no-one staying at the cottage and arbitrary conditions correspond to no discomfort.

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Fig. 1. List of possible values for objects and attributes in a heterogeneous case.

- Note that hot water, $\text{water}_{\text{Eva}} \ d = 0$ for all $d \in D_{\text{Eva}}$, because the availability of hot water and an arbitrary number of days correspond to no discomfort.
- Note that in + tv $\text{services}_{\text{John}} \ d = 0$ for all $d \in D_{\text{John}}$, because the presence of all services and an arbitrary number of days correspond to no discomfort.
- The following demonstrates monotonicity: staying on Saturday and cold water represent partial discomfort for John, but 2 days and cold water lead to large discomfort.

Fig. 2. Isotone and left-continuous operations.
Similarly, staying on 1 day and internet only represent partial discomfort for Eva, but 2 days and internet only or missing internet lead to large discomfort.

Missing TV and Saturday represent one-third discomfort for John; missing internet and Saturday, two-thirds discomfort; and missing internet and Sunday or 2 days, large discomfort.

The following demonstrates left-continuity: Saturday or Sunday and internet only represent one-third discomfort for John, but the supremum of these days (Saturday+Sunday) and internet only also lead to one-third discomfort.

Each person defines the acceptable degree of discomfort for specific conditions: i.e., \( R(\text{water, Eva}) \in P(\text{water, Eva}) \), \( R(\text{services, Eva}) \in P(\text{services, Eva}) \), \( R(\text{water, John}) \in P(\text{water, John}) \) and \( R(\text{services, John}) \in P(\text{services, John}) \).

For example, Eva can accept large discomfort for water conditions and partial discomfort for services, as shown in Fig. 2. Partial discomfort for water conditions and partial discomfort for services, as shown in Fig. 2.

Large discomfort for water conditions (value of 1) for Eva means that she will accept an arbitrary number of days and hot or cold water, because all values for water for Eva are less than or equal to 1 in Fig. 2. Partial discomfort for services (value of 1/2) for Eva means that she will accept neither all services nor a maximum of one arbitrary day with internet only, because the values for these cases for Eva in Fig. 2 are less than or equal to 1/2.

To apply FCA and identify the cottage conditions that fulfill all personal requirements, we define the mappings \( \nearrow \) and \( \searrow \).

Let \( F \) be a set of all functions \( f \) with a domain \( A \) such that \( f(a) \in C_a \) for all \( a \in A \) (more formally, \( F = \Pi_{a \in A} C_a \)).

Let \( G \) be a set of all functions \( g \) with a domain \( B \) such that \( g(b) \in D_b \) for all \( b \in B \) (i.e., \( G = \Pi_{b \in B} D_b \)).

We define the mapping \( \nearrow : G \to F \). If \( g \in G \), then \( \nearrow (g) \in F \) is defined by

\[
(\nearrow (g))(a) = \sup \{ c \in C_a : (\forall b \in B) c \bullet_{a,b} g(b) \leq R(a, b) \}.
\]

Symmetrically, we define the mapping \( \searrow : F \to G \). If \( f \in F \), then \( \searrow (f) \in G \) is defined as

\[
(\searrow (f))(b) = \sup \{ d \in D_b : (\forall a \in A) f(a) \bullet_{a,b} d \leq R(a, b) \}.
\]

(\( \nearrow \), \( \searrow \)) are defined by applying the so-called (sup, \( \bullet_{a,b} \))-product. The properties of such a structure are described in the literature [4,10].

**Theorem 1.** Let \( f \in F \) and \( g \in G \). Then the following conditions are equivalent:

1. \( f \leq \nearrow (g) \).
2. \( g \leq \searrow (f) \).
3. \( f(a) \bullet_{a,b} g(b) \leq R(a, b) \) for all \( a \in A \) and \( b \in B \).

**Proof.**

1 \( \rightarrow \) 3 Let \( a \in A \) and \( b \in B \). Then \( f \leq \nearrow (g) \) implies that \( f(a) \leq (\nearrow (g))(a) \). Hence, by the definition of \( \nearrow \) we have

\[
f(a) \leq \sup \{ c \in C_a : (\forall b' \in B) c \bullet_{a,b'} g(b') \leq R(a, b') \}.
\]

Because the condition \( (\forall b' \in B) c \bullet_{a,b'} g(b') \leq R(a, b') \) implies the condition \( c \bullet_{a,b} g(b) \leq R(a, b) \) for \( b \) (as a special case of \( b' = b \)), we have

\[
\{ c \in C_a : (\forall b' \in B) c \bullet_{a,b'} g(b') \leq R(a, b') \} \subseteq \{ c \in C_a : c \bullet_{a,b} g(b) \leq R(a, b) \},
\]

so

\[
f(a) \leq \sup \{ c \in C_a : (\forall b' \in B) c \bullet_{a,b'} g(b') \leq R(a, b') \} \leq \sup \{ c \in C_a : c \bullet_{a,b} g(b) \leq R(a, b) \}.
\]

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By the isotony of $\bullet_{a,b}$ in the first argument, we obtain
\[ f(a) \bullet_{a,b} g(b) \leq \sup\{c \in C_a : c \bullet_{a,b} g(b) \leq R(a,b)\} \bullet_{a,b} g(b). \]

Let $X$ denote the supremised set on the right-hand side of our inequality. Then $c \bullet_{a,b} g(b) \leq R(a,b)$ for every $c \in X$. Hence, by the left-continuity of $\bullet_{a,b}$ in the first argument, we have $\sup X \bullet_{a,b} g(b) \leq R(a,b)$. However, from the transitivity of $\leq$ we obtain $f(a) \bullet_{a,b} g(b) \leq R(a,b)$.

2 $\rightarrow$ 3 This can be proved analogously.

3 $\rightarrow$ 1 Let $(\forall b \in B) f(a) \bullet_{a,b} g(b) \leq R(a,b)$ hold for all $a \in A$. This means that $f(a)$ is an element of the set 
\[ \{c \in C_a : (\forall b \in B) c \bullet_{a,b} g(b) \leq R(a,b)\} \]
and hence
\[ f(a) \leq \sup\{c \in C_a : (\forall b \in B) c \bullet_{a,b} g(b) \leq R(a,b)\} = (\mathcal{R}(g))(a). \]

Because $a$ is arbitrary, we have $f \leq \mathcal{R}(g)$.

3 $\rightarrow$ 2 This can be proved analogously. \(\square\)

**Corollary 1.** Mappings $\mathcal{R}$ and $\mathcal{L}$ form a Galois connection.

**Proof.** The proof follows from the equivalence of conditions 1 and 2 of the previous theorem. \(\square\)

**Corollary 2.**

1a) $g_1 \leq g_2$ implies $\mathcal{R}(g_1) \geq \mathcal{R}(g_2)$.

1b) $f_1 \leq f_2$ implies $\mathcal{L}(f_1) \geq \mathcal{L}(f_2)$.

2a) $g \geq \mathcal{L}(\mathcal{R}(g))$.

2b) $f \leq \mathcal{R}(\mathcal{L}(f))$.

3a) $(\mathcal{L}(g)) = (\mathcal{R}(\mathcal{L}(\mathcal{R}(g))))$.

3b) $(\mathcal{R}(f)) = (\mathcal{R}(\mathcal{R}(\mathcal{L}(f))))$.

**Proof.** The proof is a consequence of $(\mathcal{R}, \mathcal{L})$ being a Galois connection [12]. \(\square\)

We use mappings $(\mathcal{R}, \mathcal{L})$ to identify the required cottage conditions as follows. Mapping $(\mathcal{L}(f))(b)$ indicates maximization of the number of days spent at the cottage for specific water and services conditions that return the greatest degree of discomfort accepted by a person. For instance, for $f(\text{water}) = \text{hot}$, $f(\text{services}) = \text{in}$ we obtain $(\mathcal{L}(f))(\text{Eva}) = 1/2$, which means that hot water and internet only correspond to a maximum stay of 1 day for Eva. Mapping $(\mathcal{R}(g))(a)$ indicates the worst water or services conditions at the cottage for a specific number of days that return the greatest degree of discomfort accepted. For instance, if $g(\text{Eva}) = 1/2$, $g(\text{John}) = \text{Sa}$, then we obtain $(\mathcal{R}(g))(\text{water}) = \text{cold}$, which means that Eva staying for 1 day and John staying on Saturday correspond to the possibility of cold water at the cottage. In another example, if $g(\text{Eva}) = \text{Sa} + \text{Su}$, $g(\text{John}) = \text{Sa}$, then we obtain $(\mathcal{R}(g))(\text{services}) = \text{in + tv}$, whereby a cottage with an internet connection and TV is the worst possible case if Eva stays on Saturday and Sunday and John stays on Saturday.

3. Heterogeneous formal concept

We use the Galois connection $(\mathcal{R}, \mathcal{L})$ for concept lattice construction via the classical Ganter–Wille approach [12].

**Lemma 1.**

1) Let $\{g_i : i \in I\} \subseteq G$. Then
\[ \mathcal{R}\left(\bigvee_{i \in I} g_i\right) = \bigwedge_{i \in I} \mathcal{R}(g_i). \]
Fig. 4. Heterogeneous formal concepts.

(2) Let \( \{ f_i : i \in I \} \subseteq F \). Then

\[
\lor \left( \bigvee_{i \in I} f_i \right) = \bigwedge_{i \in I} \lor(f_i).
\]

**Proof.** The proof is a consequence of \((\nearrow, \searrow)\) being a Galois connection [12].

We call a pair \( \langle g, f \rangle \) from \( G \times F \) such that \( \nearrow(g) = f \) and \( \searrow(f) = g \) a concept.

**Lemma 2.** If \( \langle g_1, f_1 \rangle \) and \( \langle g_2, f_2 \rangle \) are concepts, then \( g_1 \leq g_2 \) iff \( f_1 \geq f_2 \).

**Proof.** The proof is a simple consequence of Corollary 2 (parts 3a and 3b).

This lemma allows us to define the following ordering of concepts: \( \langle g_1, f_1 \rangle \leq \langle g_2, f_2 \rangle \) iff \( g_1 \leq g_2 \) (or equivalently \( f_1 \geq f_2 \)).

In summary, five concepts are obtained in our example, as shown in Fig. 4.

Intents correspond to the worst cottage conditions that fulfill all personal requirements for a specific number of days. For example, cold water and no services at the cottage correspond to no stay for Eva and staying on Saturday for John (first concept). By contrast, hot water and full services correspond to the maximum number of days spent at the cottage for both Eva and John (last concept). The next three concepts can be interpreted similarly. For instance, for the third concept, stays of 1 day by Eva and 2 days by John correspond to hot water and an internet connection as the worst possible conditions.

Note that intents do not include the possibility of hot water and no services simultaneously. In this case we obtain \( \searrow(\text{hot, no}) = (\emptyset, \text{Sa}) \) and subsequently \( \nearrow(\emptyset, \text{Sa}) = (\text{cold, no}) \). This can be interpreted as superfluous conditions for John’s stay on Saturday and maybe a cheaper cottage can be chosen.

All computations in our cottage example are for two people, but it is possible to consider more complex examples, as in Fig. 1. This case also contains the poset \( P_{\text{services, Ken}} \), where \( P_{\text{services, Ken}} = \{0, \text{le, se, 1}\} \) denotes comfort, discomfort for length of stay, discomfort for services and large discomfort, respectively. In such cases it is possible for two people to have the same lattice structure (e.g., Eva and Lea). Nevertheless, the behavior of Eva and Lea can differ for different conditions. A stay of 1 day and cold water could correspond to discomfort for Eva but comfort for Lea or vice versa.

In conclusion, this heterogeneous proposal on a formal context works with heterogeneous structures for objects, attributes and table values, so it is not possible to express this using existing approaches.

### 4. Heterogeneous formal concept lattice

We call the poset of all concepts ordered by \( \leq \) a heterogeneous concept lattice, denoted by \( \text{HCL}(A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot, \searrow, \nearrow, \leq) \).

The following theorem shows that this is in reality a lattice. Analogous theorems have been proved for existing approaches [11,12,15]. To unify these and show that the same conditions are fulfilled for our high level of generalization, the proof is self-contained.
Theorem 2 (Basic theorem on heterogeneous concept lattices).

(1) A heterogeneous concept lattice $HCL(A, B, \mathcal{P}, R, C, D, \sqcup, \mathcal{A}, \sqcap)$ is a complete lattice in which

$$\bigwedge_{i \in I} (g_i, f_i) = \left\{ \bigwedge_{i \in I} g_i, \mathcal{A} \left( \bigvee_{i \in I} f_i \right) \right\}$$

and

$$\bigvee_{i \in I} (g_i, f_i) = \left\{ \mathcal{A} \left( \bigvee_{i \in I} g_i \right), \bigwedge_{i \in I} f_i \right\}.$$ 

(2) For each $a \in A$ and $b \in B$, let $P_{a,b}$ have the least element $0_{P_{a,b}}$ such that $0_{C_a} \bullet_{a,b} d = c \bullet_{a,b} 0_{D_b} = 0_{P_{a,b}}$ for all $c \in C_a$, $d \in D_b$. Then a complete lattice $L$ is isomorphic to $HCL(A, B, \mathcal{P}, R, C, D, \sqcup, \mathcal{A}, \sqcap)$ if and only if there are mappings $\alpha : \bigcup_{a \in A} \{(a) \times C_a \} \to L$ and $\beta : \bigcup_{b \in B} \{(b) \times D_b \} \to L$ such that:

1a) $\alpha$ does not increase in the second argument (for a fixed first argument);

1b) $\beta$ does not decrease in the second argument (for a fixed first argument);

2a) $\text{Rng}(\alpha)$ is inf-dense in $L$;

2b) $\text{Rng}(\beta)$ is sup-dense in $L$; and

(3) For every $a \in A$, $b \in B$, $c \in C_a$ and $d \in D_b$,

$$\alpha(a, c) \geq \beta(b, d) \quad \text{if and only if} \quad c \bullet_{a,b} d \leq R(a, b).$$

Proof.

(1) We prove only the first part, since proof of the second part is analogous.

Using Lemma 1 we obtain

$$\mathcal{A} \left( \bigwedge_{i \in I} g_i \right) = \mathcal{A} \left( \bigwedge_{i \in I} \mathcal{A}(f_i) \right) = \mathcal{A} \left( \bigvee_{i \in I} f_i \right).$$

Using Corollary 1 (part 3b) and the fact that $(g_i, f_i)$ is a concept, we have

$$\mathcal{A} \left( \bigvee_{i \in I} f_i \right) = \bigwedge_{i \in I} \mathcal{A}(f_i) = \bigwedge_{i \in I} g_i.$$ 

This proves that $(\bigwedge_{i \in I} g_i, \mathcal{A}(\bigvee_{i \in I} f_i))$ is a concept.

Clearly $(\bigwedge_{i \in I} g_i)(b) = \bigwedge_{j \in j \in I i} g_j (b) \leq g_i (b)$, which means that $\bigwedge_{j \in I i} g_j \leq g_i$ for each $i \in I$. Thus, our concept is the lower bound of all $(g_i, f_i)$.

We show that this is the greatest. Let $(g', f')$ be some other lower bound. This means that $g' \leq g_i$ for all $i \in I$, and hence $g' \leq \bigwedge_{i \in I i} g_i$. However, this means that $(g', f') \leq (\bigwedge_{i \in I i} g_i, \mathcal{A}(\bigvee_{i \in I i} f_i))$.

(2) First we consider a special case, the lattice $HCL(A, B, \mathcal{P}, R, C, D, \sqcup, \mathcal{A}, \sqcap)$ or $H$ in short, and show the implication $\Rightarrow$.

We define the following singleton functions $S_a^c \in F$ for each $a \in A$ and $c \in C_a$, and $T_b^d \in G$ for each $b \in B$ and $d \in D_b$:

$$S_a^c(x) = \begin{cases} c & \text{if } x = a, \\ 0_{C_a} & \text{elsewhere} \end{cases} \quad \text{and} \quad T_b^d(y) = \begin{cases} d & \text{if } y = b, \\ 0_{D_b} & \text{elsewhere}. \end{cases}$$

Note that

$$(\mathcal{A}(S_a^c))(b) = \sup\{d \in D_b : c \bullet_{a,b} d \leq R(a, b)\}$$

1 Rng$(\alpha)$ denotes the mapping range of $\alpha$. 

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because by the definition of \( S_a' \), for each \( a' \neq a \),

\[
S_a'(a') \cdot_{a',b} d = 0_{C_a'} \cdot_{a',b} d = 0_{\mathcal{P}_{a',b}},
\]

and hence the condition \( S_a'(a') \leq R(a', b) \) is automatically fulfilled. Dually,

\[
(\mathcal{N}(T_b^d))(a) = \sup\{c \in C_a : c \cdot_{a,b} d \leq R(a, b)\}.
\]

Now let \( \alpha_H(a, c) = \langle \mathcal{N}(S_a'), \mathcal{N}(S_a') \rangle \)

and

\[
\beta_H(b, d) = \langle \mathcal{N}(T_b^d), \mathcal{N}(T_b^d) \rangle.
\]

Note that both \( \alpha_H(a, c) \) and \( \beta_H(b, d) \) are concepts (by Corollary 2) and hence it is enough to consider their first coordinates, for example.

Now we prove that they have the desired properties.

(1a) Let \( a \in A, b \in B, c_1, c_2 \in C_a \) and \( c_1 \leq c_2 \). By the isotony of \( \cdot_{a,b} \) in the first argument, we have \( c_1 \cdot_{a,b} d \leq c_2 \cdot_{a,b} d \) for all \( d \in D_b \). This means that

\[
\{d \in D_b : c_1 \cdot_{a,b} d \leq R(a, b)\} \supseteq \{d \in D_b : c_2 \cdot_{a,b} d \leq R(a, b)\},
\]

from which it follows that

\[
(\mathcal{N}(S_a'))(b) = \sup\{d \in D_b : c_1 \cdot_{a,b} d \leq R(a, b)\}
\]

\[
\sup\{d \in D_b : c_2 \cdot_{a,b} d \leq R(a, b)\} = (\mathcal{N}(S_a'^2))(b)
\]

for all \( b \in B \). This means that \( \mathcal{N}(S_a') \geq \mathcal{N}(S_a'^2) \), which implies \( \alpha_H(a, c_1) \geq \alpha_H(a, c_2) \).

(1b) This can be proved dually.

(2a) We prove that if \( \langle g, f \rangle \) is a concept, then \( \langle g, f \rangle = \inf\{\alpha_H(a, f(a)) : a \in A\} \), which implies the required assertion.

Let \( a \in A \). Then \( S_a f(a) = f(a) \) and \( S_a f(a') = 0_{C_a} \leq f(a') \) for all \( a' \neq a \), so \( S_a f(a) \leq f \). By Corollary 2, we obtain \( \mathcal{N}(S_a f(a)) \geq \mathcal{N}(f) = g \), which means that \( \langle g, f \rangle \leq \alpha_H(a, f(a)) \) (both are concepts, so it is enough to consider their first coordinates). This is true for all \( a \in A \), and hence \( \langle g, f \rangle \leq \inf\{\alpha_H(a, f(a)) : a \in A\} \).

Let \( \langle g', f' \rangle \) be some other lower bound of \( \{\alpha_H(a, f(a)) : a \in A\} \). Then for each \( a \in A \) we have \( \langle g', f' \rangle \leq \alpha_H(a, f(a)) \), so \( g' \leq \mathcal{N}(S_a f(a)) \) (these are their first coordinates). By Theorem 1, we have \( S_a f(a) \cdot_{a,b} g' \leq R(a, b) \), that is, \( f(a) \cdot_{a,b} g'(b) \leq R(a, b) \) for all \( a \in A \). This means that

\[
g'(b) \in \{d \in D_b : (\forall a \in A) f(a) \cdot_{a,b} d \leq R(a, b)\},
\]

so

\[
g'(b) \leq \sup\{d \in D_b : (\forall a \in A) f(a) \cdot_{a,b} d \leq R(a, b)\}
\]

\[
= (\mathcal{N}(f))(b) = g(b)
\]

for all \( b \in B \), that is, \( g' \leq g \) and hence \( \langle g', f' \rangle \leq \langle g, f \rangle \). However, this means that \( \langle g, f \rangle = \inf\{\alpha_H(a, f(a)) : a \in A\} \). Thus, each concept can be written as the infimum of some elements from \( \text{Rng}(\alpha_H) \), that is, the set is inf-dense.

(2b) It can be proved dually that \( \langle g, f \rangle = \sup\{\beta(b, g(b)) : b \in B\} \), which implies the required assertion.

(3) \( c \cdot_{a,b} d \leq R(a, b) \) means that \( c \in \{c' \in C_a : c' \cdot_{a,b} d \leq R(a, b)\} \), which implies that \( S_a'(a) = c \leq \sup\{c' \in C_a : c' \cdot_{a,b} d \leq R(a, b)\} = (\mathcal{N}(T_b^d))(a) \). For \( a' \neq a \) we also have \( S_a'(a') = 0_{C_a} \leq (\mathcal{N}(T_b^d))(a') \). Hence, \( S_a' \leq \mathcal{N}(T_b^d) \). By Corollary 2, we obtain

\[
\mathcal{N}(S_a') \geq \mathcal{N}(\mathcal{N}(T_b^d)),
\]

which means that \( \alpha_H(a, c) \geq \beta_H(b, d) \) (both are concepts, so it is enough to consider their first coordinates).
Claim 1.

(a) \( \mathcal{N}(g_\ell) = f_\ell; \)

(b) \( \mathcal{V}(f_\ell) = g_\ell. \)

Proof.

(a) By Theorem 1 it is enough to prove \( f_\ell(a) \bullet_{a,b} g_\ell(d) \leq R(a, b) \) for each \( a \in A \) and \( b \in B. \) Let \( c \in C_a \) be such that \( c \in \{ c' \in C_a : z(a, c) \geq \ell \}, \) that is, \( z(a, c) \geq \ell. \) Similarly, let \( d \in D_b \) be such that \( d \in \{ d' \in D_b : \beta(b, d) \geq \ell \}, \) that is, \( \beta(b, d) \leq \ell. \) By the transitivity of \( \leq \) we obtain \( z(a, c) \geq \beta(b, d), \) which means (by the assumptions for \( z \) and \( \beta \)) that \( c \bullet_{a,b} d \leq R(a, b). \)

(b) This can be proved analogously. \( \square \)
Now we show that $\zeta$ is the required isomorphism and $\psi$ is its inverse.

**Claim 2.** $\zeta$ preserves the ordering.

**Proof.** Let $\langle g_1, f_1 \rangle \leq \langle g_2, f_2 \rangle$ be two concepts. Then, by definition, $f_1 \geq f_2$. Because $\alpha$ is non-increasing in the second argument, we obtain $\alpha(a, f_1(a)) \leq \alpha(a, f_2(a))$ for each $a \in A$. It follows that

$$\zeta(g_1, f_1) = \inf \{\alpha(a, f_1(a)) : a \in A\} \leq \inf \{\alpha(a, f_2(a)) : a \in A\} = \zeta(g_2, f_2).$$

□

**Claim 3.** $\psi$ preserves the ordering.

**Proof.** Let $\ell_1 \leq \ell_2$. It follows from transitivity that

$$\{c \in C_a : \alpha(a, c) \geq \ell_1\} \supseteq \{c \in C_a : \alpha(a, c) \geq \ell_2\}$$

and hence

$$f_{\ell_1}(a) = \sup \{c \in C_a : \alpha(a, c) \geq \ell_1\} \geq \sup \{c \in C_a : \alpha(a, c) \geq \ell_2\} = f_{\ell_2}(a)$$

for each $a \in A$. This means that $f_{\ell_1} \geq f_{\ell_2}$, so

$$\psi(\ell_1) = \langle g_{\ell_1}, f_{\ell_1} \rangle \leq \langle g_{\ell_2}, f_{\ell_2} \rangle = \psi(\ell_2).$$

□

**Claim 4.** $\bar{\zeta}(\psi(\ell)) = \ell$.

**Proof.** Because of the sup-density of $\text{Rng}(\beta)$, $\ell$ can be written as the supremum of some set of pairs from $\text{Rng}(\beta)$, that is, $\ell = \sup \{\beta(b_i, d_i) : i \in I\}$, where $d_i \in D_{b_i}$ for each $i \in I$. This means that $\beta(b_i, d_i) \leq \ell$ for each $i \in I$.

If $a \in A$ and $c \in C_a$ are arbitrary such that $\alpha(a, c) \geq \ell$, we obtain $\alpha(a, c) \geq \beta(b_i, d_i)$ for each $i \in I$. By property (3) for $\alpha$ and $\beta$, we obtain $c \cdot_{a,b_i} d_i \leq R(a, b_i)$. Hence, by the left-continuity of $\cdot_{a,b_i}$ in the first argument, we have

$$f_{\ell}(a) \cdot_{a,b_i} d_i = \sup \{c \in C_a : \alpha(a, c) \geq \ell\} \cdot_{a,b_i} d_i \leq R(a, b_i)$$

for each $i \in I$. Again by Assumption (3) we have $\alpha(a, f_{\ell}(a)) \geq \beta(b_i, d_i)$ for each $i \in I$, and hence

$$\alpha(a, f_{\ell}(a)) \geq \sup \{\beta(b_i, d_i) : i \in I\} = \ell.$$

It follows that

$$\bar{\zeta}(\psi(\ell)) = \bar{\zeta}(\langle g_{\ell}, f_{\ell} \rangle) = \inf \{\alpha(a, f_{\ell}(a)) : a \in A\} \geq \ell.$$

Conversely, because of the inf-density of $\text{Rng}(\alpha)$, $\ell$ can be written as the infimum of some set of pairs from $\text{Rng}(\alpha)$, that is, $\ell = \inf \{\alpha(a_i, c_i) : i \in I\}$, where $c_i \in C_{a_i}$ for each $i \in I$. Obviously $\alpha(a_i, c_i) \geq \ell$, so $c_i \in \{c \in C_{a_i} : \alpha(a_i, c) \geq \ell\}$, from which it follows that

$$c_i \leq \sup \{c \in C_{a_i} : \alpha(a_i, c) \geq \ell\} = f_{\ell}(a_i)$$

for each $i \in I$. Because $\alpha$ is non-increasing in the second argument, we obtain

$$\alpha(a_i, c_i) \geq \alpha(a_i, f_{\ell}(a_i)) \geq \inf \{\alpha(a, f_{\ell}(a)) : a \in A\} = \bar{\zeta}(\langle g_{\ell}, f_{\ell} \rangle) = \bar{\zeta}(\psi(\ell))$$

for each $i \in I$. This means that $\ell = \inf \{\alpha(a_i, c_i) : i \in I\} \geq \bar{\zeta}(\psi(\ell))$, so the opposite inequality is also proven. □

**Claim 5.** $\psi(\bar{\zeta}(\langle g, f \rangle)) = \langle g, f \rangle$.

**Proof.** We denote $\ell = \bar{\zeta}(\langle g, f \rangle)$. We want to prove $\psi(\ell) = \langle g, f \rangle$, that is, $\langle g_{\ell}, f_{\ell} \rangle = \langle g, f \rangle$. Because both are concepts, it is enough to prove $g_{\ell} = g$. 

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Using property (3), we obtain \((\forall a \in A) f(a) \bullet_{a,b} d \leq R(a, b)\) if and only if \((\forall a \in A) g(b, d) \leq \zeta(a, f(a))\). By the definition of the infimum, this is equivalent to \(g(b, d) \leq \inf \{\zeta(a, f(a)) : a \in A\} = \zeta((g, f)) = \ell\). This means that
\[
\{d \in D_b : (\forall a \in A) f(a) \bullet_{a,b} d \leq R(a, b)\} = \{d \in D_b : g(b, d) \leq \ell\},
\]
from which it follows that
\[
(\mathcal{N}(f))(b) = \sup \{d \in D_b : (\forall a \in A) f(a) \bullet_{a,b} d \leq R(a, b)\} = \sup \{d \in D_b : g(b, d) \leq \ell\} = g_\ell(b),
\]
that is, \(g = \mathcal{N}(f) = g_\ell\).

We have proved that both \(\zeta\) and \(\psi\) preserve the ordering and both their compositions are identities. This means that \(\zeta = \psi^{-1}\) and both are isomorphisms.

5. Generalization

Our previous approach called a generalized (fuzzy) concept lattice [13,14] is obviously a special case of this new approach; it is enough to take the same \(D_a, D_b, \leq_{a,b}\) and \(\bullet_{a,b}\).

The multi-adjointness approach of Medina and Ojeda-Aciego [17] (that generalizes the approach from [19] in which the adjoint triples are defined on \((L,L,P)\)) has some rather strange and not very aesthetic property: it takes only \(g_b\) for \(b \in B\) without any reference to \(A\). We symmetrize it and take operations \(\bullet_{a,b}\) for each pair \((a, b) \in A \times B\). This, of course, diversifies and generalizes [17]. Moreover we need not the equal lattices for all \(b \in B\) and/or all \(a \in A\).

We note one issue. The approach of Medina and Ojeda-Aciego [17] works with adjoints to \(\bullet\) operations. Recall the notion: Let \((C, \leq_C), (D, \leq_D)\) and \((P, \leq_P)\) be posets. The triple \((\bullet, \rightarrow_1, \rightarrow_2)\) is called an adjoint triple if \(\bullet : (C \times D) \rightarrow P, \rightarrow_1 : (D \times P) \rightarrow C, \rightarrow_2 : (C \times P) \rightarrow D\) and
\[
(c \bullet d) \leq_P \text{ iff } c \leq (d \rightarrow_1 p) \text{ iff } d \leq (c \rightarrow_2 p).
\]

(If \(C = D\) and \(\bullet\) is commutative, both arrows are identical.) Hence, all the substantial definitions of Medina and Ojeda-Aciego [17] can be reformulated using \(\bullet\) operations only.

For each operation \(\bullet\) that is isoton and left-continuous in both arguments, there are operations \(\rightarrow_1\) and \(\rightarrow_2\) such that \((\bullet, \rightarrow_1, \rightarrow_2)\) is an adjoint triple; it is enough to define
\[
d \rightarrow_1 p = \sup \{c \in C : c \bullet d \leq_P p\}
\]
and symmetrically
\[
c \rightarrow_2 p = \sup \{d \in D : c \bullet d \leq_P p\}.
\]
Obviously, \(c \bullet d \leq_P p\) means that \(c \in \{c' \in C : c' \bullet d \leq_P p\}\) and hence \(c \leq_C \sup \{c' \in C : c' \bullet d \leq_P p\} = d \rightarrow_1 p\).

Conversely, by the left-continuity of \(\bullet\) in the first argument, we have \(\sup \{c' \in C : c' \bullet d \leq_P p\} \bullet d \leq_P p\), that is, \((d \rightarrow_1 p) \bullet d \leq_P p\). Thus, if \(c \leq_C d \rightarrow_1 p\), then by the isotonity of \(\bullet\) in the first argument we have \(c \bullet d \leq_P p\).

The dual properties of \(\rightarrow_2\) can be proved symmetrically. This means that working with adjoint triples is equivalent to working with isoton and left-continuous functions.

6. Conclusions and future work

We introduced a new common platform for fuzzification of FCA. The main idea is diversification of all that can be diversified. The main advantage of our approach is removal of the homogeneity barrier. It intuitively follows that FCA could be used for tables with data of different types. We will consider these issues in further research.

Pócs made an alternative attempt to cover all known approaches [23,24]. This also works with different lattices but it gives not just a single value but a Galois connection to fields of the matrix (i.e., certain behaviors for relationship between the object and the attribute). However, it seems rather cumbersome to work with whole Galois connections in single data fields. Our approach has no limitation regarding values in data fields. We have previously compared these approaches [1].
Another interesting issue would be to clarify the relationship of this approach to the fuzzification of Bělohlávek and Vychodil for truth-stressers or so-called hedges [6,7]. Krajčí showed that generalized concept lattices cover these in some sense [16], but our new approach seems to make this relationship more immediate.

Another interesting relationship for future work is heterogeneity in multi-adjoint concept lattices that are based on heterogeneous conjunctors [18].

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