Centralized Communication in Radio Networks with Strong Interference

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Abstract. We study communication in known topology radio networks with the presence of interference constraints. We consider a real-world situation, when a transmission of a node produces an interference in the area that is larger than the area, where the transmitted message can be received. For each node, there is an area, where a signal of its transmission is too low to be decoded by a receiver, but is strong enough to interfere with other incoming simultaneous transmissions. Such a setting is modelled by a newly proposed interference reachability graph that extends the standard graph model based on reachability graphs. Further, focusing on the information dissemination problem in bipartite interference reachability graphs, we introduce interference ad-hoc selective families as an useful combinatorial tool. They are a natural generalization of ad-hoc selective families. Adopting known algorithms and techniques, we show how to construct small interference ad-hoc selective families in the case when, for each node, the ratio of the only-interfering neighbors to the other neighbors is bounded. Finally, taking into account the maximum degree in an underlying interference reachability graph, we study the broadcasting problem in general radio networks.

1 Introduction

A *radio network* is a collection of autonomous stations that are referred to as *nodes*. The nodes communicate via sending messages. Each node is able to receive and transmit messages. However, a node can transmit messages only to the nodes, which are located within its *transmission range*. We say that a node w belongs to the transmission range of a node v ($w \in T(v)$) if and only if a message transmitted by v can reach the node w. Hence, the transmission range of v is a set of the network nodes that are located at positions, where the signal transmitted by v has enough intensity and quality to be successfully decoded. All nodes of the radio network operate at the same frequency. Owing to properties of the radio communication medium, simultaneous transmissions of two or more nodes cause interference in the area that is in the range of those transmitted

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signals. However in some practical applications, a transmitted signal can reach an area, where decoding of the signal is not possible due to its low intensity, but the signal is intensive enough to interfere with other simultaneous transmissions. We define an *interference range* $I(v)$ of a node v as follows. A node w belongs to the interference range of a node v ($w \in I(v)$) if and only if a transmission of v can interfere with other transmissions reaching the node w. It is natural to assume that $T(v) \subseteq I(v)$. Indeed, if a signal is intensive enough to be decoded, it is intensive enough to cause interference with other transmissions.

Most of the literature concerning communication in radio networks (e.g. [10], [5], [9]) assume that the transmission range $T(v)$ and the interference range $I(v)$ of a node v are the same, i.e. $T(v) = I(v)$ for each network node v. Such a communication network can be modelled by a *reachability graph.* A reachability graph is a directed graph $G = (V, E)$. The vertex set V corresponds to the network nodes and two vertices $u, v \in V$ are connected by an edge $e = (u, v)$ if and only if the node v is in the transmission range of a node u, i.e. $v \in T(u)$. This model is also referred to as the *graph model* of radio networks. If the transmission power of all nodes is the same, then a network can be modelled by an undirected graph.

A communication model, in which the interference range of a node is larger than its transmission range, was considered by Bermond et al. in [2]. The authors studied time complexity of the gathering task in known topology radio networks. They defined transmission and interference range of a node with respect to distances in an underlying communication graph. Particularly, denote by $dist_G(u, v)$ the length (the number of edges) of a shortest path between nodes u and v in the graph G. Fix the numbers d_T and d_I. The number d_T, $d_T \geq 1$, is called a *transmission distance* and the number d_I, $d_I \geq d_T$, is called an *interference distance*. The transmission range $T(v)$ of a node v is defined as $T(v) = \{w | dist(v, w) \leq d_T\}$ and the interference range $I(v)$ of a node v as $I(v) = \{w | dist(v, w) \leq d_I\}$. Note that the standard graph model corresponds to the case when $d_T = d_I = 1$.

It is easy to see that there are such settings, where the model introduced by Bermond et al. is not appropriate, e.g. due to large obstacles or signal reflexes. That means, that there are settings, for which it is difficult or even impossible to express the transmission and interference ranges of the nodes with respect to distances in an underlying communication graph. In this paper we focus on a new model of the communication environment. Particularly, we shall assume that $I(v)$ is an arbitrary set of network nodes satisfying $T(v) \subseteq I(v)$. In such a setting, the communication network can be modelled by a directed graph $G = (V, E_T \cup E_I)$, called an *interference reachability graph* (IRG), such that $E_T \cap E_I = \emptyset$. Two vertices $u, v \in V$ are connected by an edge $e = (u, v) \in E_T$ (a *transmission edge*) if and only if the node v is in the transmission range of the node u, i.e. $v \in T(u)$. The node u is referred to as a *transmission neighbor* of the node v. Similarly, two vertices $u, v \in V$ are connected by an edge $e = (u, v) \in E_I$ (an *interference edge*) if and only if the node v is in the interference range of the node u but not in its transmission range, i.e. $v \in I(u) \setminus T(u)$. The node u is referred to as an *interference neighbor* of the node v. Note that no message can be brought forward by an interference edge in compare to a transmission edge. We shall denote the spanning subgraph $G(E_T)$ as a *transmission subgraph*. For practical reasons, we assume that the transmission subgraph $G(E_T)$ is strongly connected. Hence there is an oriented path, using only transmission edges, from each network node to any other network node. Observe that each radio network modelled by the model introduced by Bermond et al. can be described by an IRG. It implies that the proposed model is more general.

Communication in radio networks is considered to be synchronous. In particular, the network nodes work in synchronized steps (time slots) called *rounds*. In every round, a node can act either as a *receiver* or as a *transmitter*. A node u acting as transmitter sends a message, which can be potentially received by every node in its transmission range. In a given round, a node, acting as a receiver, receives a message if and only if it is located in the transmission range of exactly one transmitting node and in the interference range of none transmitting node. Otherwise, no message is received by the receiving node. The received message is the same as the message transmitted by the transmitting neighbor.

2 Centralized Broadcasting

We focus on the *broadcasting* - one of the most studied and important communication primitives. The goal of broadcasting is to distribute a message, called a *source message*, from one distinguished node, called a *source*, to all other nodes. Remote nodes of the network are informed via intermediate nodes. The time (number of rounds), that is required to complete an operation, is an important efficiency measure and is a widely studied parameter of mostly every communication task.

It is known that assumptions about initial knowledge of nodes significantly influence the time required to complete a communication task. In this paper we shall assume that each node possesses full knowledge of the network topology, i.e. every node possesses a labelled copy of an underlying IRG. Communication in radio networks with full knowledge of nodes (known topology radio networks) is referred to as *centralized communication*. In this setting, an execution of a broadcasting algorithm can be seen as a process controlled by a central monitor. Thus the goal is to design a polynomial time (tractable) algorithm, that for a given (interference) reachability graph G and a source node s produces a schedule of transmissions, referred to as a *radio broadcast schedule*, that is as short as possible and disseminates the source message to all network nodes.

Centralized broadcasting in radio networks modelled by a reachability graph, i.e. in the standard graph model, has been intensively studied. In [9], Kowalski and Pelc presented an algorithm that produces a radio broadcast schedule of the length $O(D + \log^2 n)$, where D is the diameter of a given underlying reachability graph and n denotes the number of nodes. In the view of the lower bound $\Omega(\log^2 n)$ for graphs with diameter 2 given by Alon et al. [1] and, a trivial lower bound D, this algorithm is asymptotically optimal.

The rest of the paper is devoted to the information dissemination problem in known topology radio networks modelled by IRGs.

2.1 Difficulty of Fast Broadcasting in IRG

It is easy to see that the broadcasting task can be always completed in $O(n)$ rounds, where n is the number of network nodes. Hence there is a radio broadcast schedule such that each node is informed at most $O(n)$ rounds after first transmission of the source node. Indeed, in each round, we select one informed node that is a transmission neighbor of at least one uninformed node. Only the selected node is allowed to transmit in this round. Thus all uninformed nodes in its transmission range receive the source message and become informed. On the other hand, there is an IRG with diameter 2 such that the broadcasting time is bounded by $\Omega(n)$ rounds. The $(2 \cdot m + 1)$-node graph $G_m = (V_m, E_T \cup E_I)$ is defined as follows:

- $V(G_m) = \{s, a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m\}$
- $E_T(G_m) = \{(s, a_i), (a_i, b_i) | 1 \leq i \leq m\}$
- $E_I(G_m) = \{(a_i, b_j) | 1 \leq i \neq j \leq m\}$

Let s to be a source of the broadcasting. Each node a_i becomes informed after first transmission of s. The node b_i can be informed only by a transmission of a_i. However, if a node a_i transmits, no other node b_j, $j \neq i$, can receive the message due to the presence of interference edges. It follows, that at least $m + 1 = \Omega(n)$ rounds are necessary to complete the broadcasting.

Therefore, in order to study the time complexity of the broadcasting task in the proposed interference model, we should consider other parameters of IRG, or introduce new appropriate parameters expressing the presence of interference edges in the IRG.

3 Interference Ad-Hoc Selective Families

Following the work of Clementi et al. [3], that is devoted to selective structures related to the standard model of radio networks, we define the notion of *interference ad-hoc selective family* and show some useful properties of it. As we will discuss later, this notion is closely related to the considered interference model of radio networks. In the case, when a considered collection of set-pairs satisfies a specific property (defined later), we show the existence of small interference ad-hoc selective families by a probabilistic argument. Finally, we design a deterministic polynomial-time algorithm that computes small interference ad-hoc selective family for a given input collection of set-pairs. Algorithms presented in this section extends the work [3] of Clementi et al.

The (interference) ad-hoc selective families are related to intensively studied combinatorial structures called *selectors* (see e.g. [6], [4], or [8]). One of their applications is in communication algorithms under the standard graph model of radio networks [7] in the case when the nodes are not aware of the network topology. The *k-selectors*, defined and investigated by Chrobak et al. in [7], can be seen as a weaker variant of interference ad-hoc selective families introduced in this section.

Definition 1. *Let* $\mathcal{F} = \{(T_1, I_1), (T_2, I_2), \ldots, (T_m, I_m)\}$ *to be a collection of set-pairs such that* $T_i \cap I_i = \emptyset$ *and* $T_i \neq \emptyset$*, for all* $i = 1, \ldots, m$*. Denote* $U(\mathcal{F}) = \bigcup_{i=1}^{m} T_i \cup I_i$*. A family* $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ *of subsets of* $U(\mathcal{F})$ *is said to be selective for* $\mathcal{F}$ *if and only if for any* (T_i, I_i) *there is a set* S_j *such that* $|T_i \cap S_i| = 1$ *and* $I_i \cap S_i = \emptyset$*. We say that the set* S_j *is selective for* (T_i, I_i)*.*

There is a relationship between interference ad-hoc selective families and the broadcasting task. To see it, suppose that a proper subset of network nodes is already informed. Initially, only the source is informed. Let V_S to be a set of informed network nodes that have an uninformed node within transmission range. Let V_R to be a set of uninformed nodes that are located within transmission ranges of informed nodes. Interference ad-hoc selective families can be utilized to construct a schedule of transmissions such that all nodes in V_R become informed by transmissions of nodes in the set V_S. Indeed, consider a collection $\mathcal{F} = \{(T_v, I_v) | v \in V_R\}$ such that $T_v = \{u \in V_S | v \in T(u)\}$ and $I_v = \{u \in V_S | v \in I(u)\}$. Note that $U(\mathcal{F}) \subseteq V_S$. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ to be an interference ad-hoc selective family for $\mathcal{F}$. Observe, that if exactly the nodes of S_i transmit in the i-th round of a schedule, all nodes in V_R become informed in at most $k = |\mathcal{S}|$ rounds. Hence it seems useful to search for small selective families. Obviously, we can always construct a selective family of the size $min\{|\mathcal{F}|, |U(\mathcal{F})|\}$ by a trivial construction. On the other hand, for any n, there is an instance $\mathcal{F}$, $|\mathcal{F}| = |U(\mathcal{F})| = n$, such that it is not possible to construct interference ad-hoc selective family of the size smaller than n. These instances correspond to the example of "slow" IRG in the section 2.1. It follows, as for the broadcasting in IRG, that a new parameter characterizing collection $\mathcal{F}$ should be introduced and considered.

Definition 2. *Let* $\mathcal{F} = \{(T_1, I_1), (T_2, I_2), \ldots, (T_m, I_m)\}$ *to be a collection of set-pairs such that* $T_i \cap I_i = \emptyset$ *and* $T_i \neq \emptyset$*, for all* $i = 1, \ldots, m$*. We say that* r *is an interference ratio of the pair* (T_i, I_i) *if and only if* $|I_i| \leq r \cdot |T_i|$*. Analogously, we say that* $r(\mathcal{F})$ *is an interference ratio of the collection* $\mathcal{F}$*, if and only if* $|I_i| \leq r(\mathcal{F}) \cdot |T_i|$*, for all* $i = 1, \ldots, m$*.*

Intuitively, the notion of the interference ratio is introduced in order to express a ratio of the interference edges to the transmission edges of a node. Now, using a probabilistic argument, we show that there are small interference ad-hoc selective families.

Theorem 1. *Let* $\mathcal{F} = \{(T_1, I_1), (T_2, I_2), \ldots, (T_m, I_m)\}$ *to be a collection of set-pairs such that* $T_i \cap I_i = \emptyset$*,* $T_i \neq \emptyset$*, and* $\Delta_{min} \leq |T_i| + |I_i| \leq \Delta_{max}$*, for all* $i = 1, \ldots, m$*. There is a family* $\mathcal{S}$ *of the size* $O((1 + r(\mathcal{F})) \cdot ((1 + \log(\Delta_{max}/\Delta_{min}))) \cdot \log |\mathcal{F}|)$ *that is selective for* $\mathcal{F}$*.*

Proof. Let us define $\mathcal{F}' = \{(T_i, I_i) \in \mathcal{F}, |T_i| + |I_i| = 1\}$. Since $T_i \neq \emptyset$, we have that $|T_i| = 1$ and $|I_i| = 0$, for all members of $\mathcal{F}'$. It is easy to see that the set $S_0 = \bigcup_{(T_i,I_i)\in\mathcal{F}'} T_i$ is selective for $\mathcal{F}'$. Therefore, in what follows we can assume that $\Delta_{min} \geq 2$.

For each $j \in \{\lceil\log\Delta_{min}\rceil, \ldots, \lceil\log\Delta_{max}\rceil\}$, consider a family $\mathcal{S}_j$ of l sets, where an unknown parameter l will be determined at the end of the proof. Each set $S \in \mathcal{S}_j$ is constructed by picking each element of $U(\mathcal{F})$ independently with the probability $1/2^j$.

Fix a pair $(T_i, I_i) \in \mathcal{F}$. Let j to be an integer such that $2^{j-1} \leq |T_i| + |I_i| < 2^j$. Consider a set $S \in \mathcal{S}_j$. Let us estimate the probability that the set S is selective for (T_i, I_i):

$$Pr[|T_i \cap S| = 1 \wedge I_i \cap S = \emptyset] = |T_i| \cdot \frac{1}{2^j} \cdot \left(1 - \frac{1}{2^j}\right)^{|T_i|+|I_i|-1}$$

$$> |T_i| \cdot \frac{1}{2^j} \cdot \left(1 - \frac{1}{2^j}\right)^{2^j} \overset{(1)}{\geq} \frac{1}{2\cdot(r(\mathcal{F})+1)} \cdot \left(1 - \frac{1}{2^j}\right)^{2^j} \overset{(2)}{\geq} \frac{1}{8\cdot(r(\mathcal{F})+1)}$$

The inequality (1) holds because $2^{j-1} \leq |T_i| + |I_i| \leq (r(\mathcal{F}) + 1) \cdot |T_i|$. The inequality (2) follows from the fact that $\left(1 - \frac{1}{t}\right)^t \geq \frac{1}{4}$, for $t \geq 2$.

The sets in $\mathcal{S}_j$ are constructed independently. Thus the probability that none of l sets of the family $\mathcal{S}_j$ is selective for (T_i, I_i) is upper-bounded by the expression

$$\left(1 - \frac{1}{8\cdot(r(\mathcal{F})+1)}\right)^l \leq e^{-\frac{l}{8\cdot(r(\mathcal{F})+1)}}$$

due to the inequality $(1-x)^y \leq e^{-x\cdot y}$, for $0 < x < 1$ and $y > 1$.

Finally, let us define a family $\mathcal{S}$ as the union of the families $\mathcal{S}_j$, for $j \in \{\lceil\log\Delta_{min}\rceil, \ldots, \lceil\log\Delta_{max}\rceil\}$. Now we estimate the probability that $\mathcal{S}$ is not selective for $\mathcal{F}$:

$$Pr[\mathcal{S} \text{ is not selective for } \mathcal{F}] \leq \sum_{(T_i,I_i)\in\mathcal{F}} Pr[\mathcal{S} \text{ is not selective for } ((T_i, I_i))]$$

$$\leq \sum_{(T_i,I_i)\in\mathcal{F}} e^{-\frac{l}{8\cdot(r(\mathcal{F})+1)}} = |\mathcal{F}| \cdot e^{-\frac{l}{8\cdot(r(\mathcal{F})+1)}}$$

It follows, that the probability of $\mathcal{S}$ not being selective for $\mathcal{F}$, is less than 1 for $l > 8 \cdot (r(\mathcal{F}) + 1) \cdot \ln|F|$. It implies the existence of an interference ad-hoc selective family $\mathcal{S}$ of the size $O((1+r(\mathcal{F}))\cdot((1+\log(\Delta_{max}/\Delta_{min})))\cdot\log|\mathcal{F}|)$. □

Using de-randomization method of conditional probabilities, we show that a selective family of the size $O((1 + r(\mathcal{F})) \cdot ((1 + \log(\Delta_{max}/\Delta_{min}))) \cdot \log|\mathcal{F}|)$ can be constructed deterministically in the polynomial time.

At first, we fix an ordering of elements of $U(\mathcal{F})$, i.e. $U(\mathcal{F}) = \{u_1, u_2, \ldots, u_n\}$. For $S \subseteq U(\mathcal{F})$, let us denote $\delta_i(S) = S \cap \{u_i, u_{i+1}, \ldots, u_n\}$. Finally, let us fix a pair $(T, I) \in \mathcal{F}$ and let Δ to be a power of 2 (i.e. $\Delta = 2^j$, for some j) such that $\Delta/2 \leq |T| + |I| < \Delta$. For a fixed set $S \subseteq \{u_1, u_2, \ldots, u_{j-1}\}$, we define the conditional probabilities

$$Y_j(S,(T,I)) = Pr[S \cup X \cup \{u_j\} \text{ is selective for } (T,I)]$$

$$N_j(S,(T,I)) = Pr[S \cup X \text{ is selective for } (T,I)]$$

where X is a subset of $\{u_{j+1}, \ldots, u_n\}$ constructed by picking each element of $\{u_{j+1}, \ldots, u_n\}$ independently at random with the probability $1/\Delta$, that is, $Pr[u_k \in X] = 1/\Delta$.

Lemma 1. *The conditional probabilities $Y_j(S,(T,I))$ and $N_j(S,(T,I))$ can be computed in $O(n)$ time.*

Utilizing those conditional probabilities, we design an algorithm that computes an interference ad-hoc selective family for a given input collection of set-pairs $\mathcal{F}$. Algorithm is based on the procedure *IASF*. It produces an interference ad-hoc selective family for a given collection $\mathcal{F} = \{(T_1, I_1), (T_2, I_2), \ldots, (T_m, I_m)\}$ satisfying the property that there is a power of 2 denoted as Δ (i.e. $\Delta = 2^j$, for some $j \geq 2$) such that the condition $\Delta/2 \leq |T_i| + |I_i| < \Delta$ is valid for all members of $\mathcal{F}$.

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Input  : Δ = 2^j, F = {(T_1, I_1), (T_2, I_2), ..., (T_m, I_m)}
Output: S = {S_1, S_2, ..., S_k}
let n to be the number of elements of U(F) = {u_1, u_2, ..., u_n};
while F ≠ ∅ do
    S ← ∅;
    for i ← 1 to n do
        Y_i ← Σ_{(T,I)∈F} Y_i(S,(T,I)) ;
        N_i ← Σ_{(T,I)∈F} N_i(S,(T,I)) ;
        if N_i < Y_i then S ← S ∪ {u_i};
    end
    F ← F \ {(T,I) ∈ F | S is selective for (T,I)};
    S ← S ∪ S;
end
return S
```

Algorithm 1. Procedure *IASF*

Theorem 2. *Let $\mathcal{F} = \{(T_1, I_1), (T_2, I_2), \ldots, (T_m, I_m)\}$ to be a collection of set-pairs such that $T_i \cap I_i = \emptyset$, $T_i \neq \emptyset$, and $\Delta_{min} \leq |T_i| + |I_i| \leq \Delta_{max}$, for all $i = 1, \ldots, m$. There is a deterministic algorithm that produces an interference ad-hoc selective family $\mathcal{S}$ of the size $O((1+r(\mathcal{F}))\cdot((1+\log(\Delta_{max}/\Delta_{min}))) \cdot \log|\mathcal{F}|)$ for the given collection $\mathcal{F}$. Computation takes polynomial time, more precisely $O((1 + \log(\Delta_{max}/\Delta_{min})) \cdot r(\mathcal{F}) \cdot (\log|\mathcal{F}|) \cdot |\mathcal{F}| \cdot |U(\mathcal{F})|^2)$.*

Proof. The goal of the procedure *IASF* is to compute an interference ad-hoc selective family for a specific subset of the input collection $\mathcal{F}$. For each $j \in \{\lceil \log \Delta_{min} \rceil, \ldots, \lceil \log \Delta_{max} \rceil\}$, the procedure *IASF* is executed with the two input parameters: $\Delta = 2^j$ and a subset of the collection $\mathcal{F}$ (denoted as $\mathcal{F}_j$), that is restricted to those set-pair (T, I) satisfying $\Delta/2 \leq |T| + |I| < \Delta$. The resulting selective collection is the union of all selective families returned by executions of *IASF*. As in the proof of the theorem 1, we focus only on j such that $j \geq 2$. Indeed, for $j = 1$ the construction of a selective set is trivial.

At first, we show that each execution of *IASF* produces a selective family of the size at most $O((1 + r(\mathcal{F}_j)) \cdot \log |\mathcal{F}_j|)$. Let us fix considered input parameters of *IASF*: $\Delta = 2^j$ and the collection $\mathcal{F}_j$. In the following, we shall analyze a single execution of the *while* loop in the procedure *IASF*. Hence, the symbols $\mathcal{F}$ and S will correspond to the variables of the algorithm. Note that the variable (collection) $\mathcal{F}$ remains unchanged during the analyzed part of the execution. It is modified only at the end of each iteration of the *while* loop.

Let W to be a set constructed by picking each element of $U(\mathcal{F})$ independently with the probability $1/\Delta$. Denote as $E(X)$ the expected number of set-pairs $(T, I) \in \mathcal{F}$ that are selected by W. Analogously, for a set Y, $Y \subseteq U(\mathcal{F})$, and an integer i, $i \geq 1$, satisfying $Y \cap \delta_i(U(\mathcal{F})) = \emptyset$, we denote as $E(X|(Y, i))$ the expected number of set-pairs $(T, I) \in \mathcal{F}$ that are selected by a random set $W_{Y,i}$. The set $W_{Y,i}$ is the union of the set Y and a set of independently (with probability $1/\Delta$) picked elements of the set $\delta_i(U(\mathcal{F}))$. Clearly, it follows from the proof of the theorem 1 that

$$E(X|(\emptyset, 1)) = E(X) \geq \frac{|\mathcal{F}|}{8 \cdot (r(\mathcal{F}) + 1)}.$$

Now we prove by induction on i that the inequality $E(X|(S, i + 1)) \geq E(X)$ is valid after i ($i \in \{0, \ldots, |U(\mathcal{F})|\}$) iterations of the *for* loop in *IASF*:

- For $i = 0$, it holds $S = \emptyset$. Since $E(X|(\emptyset, 1)) = E(X)$, the claim is true.
- Suppose that the claim is true for all j, $j < i$. Recall that the symbol S corresponds to the set variable S (containing a subset of $U(\mathcal{F})$) in the procedure *IASF* after i iterations of the *for* loop. Denote $S' = S \setminus \{u_i\}$. Due to the definition of the expected value, it holds for $i > 0$ that

  $$E(X|(S', i)) = \frac{1}{\Delta} E(X|(S' \cup \{u_i\}, i + 1)) + \big(1 - \frac{1}{\Delta}\big) E(X|(S', i + 1)).$$

  Obviously, $A = qB + (1 - q)C \Rightarrow A \leq max\{B, C\}$, for $A, B, C \geq 0$ and $0 \leq q \leq 1$. Thus it follows

  $$E(X|(S', i)) \leq max\{E(X|(S' \cup \{u_i\}, i + 1)), E(X|(S', i + 1))\}.$$

  Moreover, the definition of the expected value implies $Y_i = E(X|(S' \cup \{u_i\}, i + 1))$ and $N_i = E(X|(S', i + 1))$. The choice, between adding the element u_i to S or not, depends on the values Y_i and N_i. Since the larger value is chosen, it follows that $E(X|(S, i + 1)) = max\{Y_i, N_i\} = max\{E(X|(S' \cup \{u_i\}, i + 1)), E(X|(S', i + 1))\} \geq E(X|(S', i))$. Finally, the inductive hypothesis implies

  $$E(X|(S, i + 1)) \geq E(X|(S', i)) \geq E(X).$$

It follows from the previous claim for $i = |U(\mathcal{F})|$, that in each iteration of the *while* loop such a set S is constructed that at least $\lceil \frac{|\mathcal{F}|}{8\cdot(r(\mathcal{F})+1)} \rceil$ set-pairs of the collection variable $\mathcal{F}$ (in execution of *IASF*) are selected by S. Thus after k iterations of the *while* loop, the number of unselected set-pairs in the collection variable $\mathcal{F}$ can be upper-bounded by the expression $\left(1-\frac{1}{8\cdot(r(\mathcal{F}_j)+1)}\right)^k \cdot |\mathcal{F}_j|$, where $\mathcal{F}_j$ is input of *IASF*. Since, for $z \geq 1$, it holds $\ln(1-\frac{1}{z}) \leq -\frac{1}{z}$, this expression is lower than 1 for k at least $(8 \cdot r(\mathcal{F}_j)+1) \cdot |\mathcal{F}_j|$. Finally, we get that at most $O((r(\mathcal{F}_j)+1)\cdot|\mathcal{F}_j|)$ iterations of the *while* loop are sufficient to select all set-pairs of the collection $\mathcal{F}_j$. Hence the interference ad-hoc selective family constructed by one execution of *IASF* has the size $O((1+r(\mathcal{F}_j)) \cdot \log|\mathcal{F}_j|)$.

There are $\lceil 1+\log(\Delta_{max}/\Delta_{min})\rceil$ executions of *IASF*. Considering the definition of the interference ratio $r(\mathcal{F})$, it is easy to see, that if $\mathcal{F}_j \subseteq \mathcal{F}$ then $r(\mathcal{F}_j) \leq r(\mathcal{F})$. This concludes the proof that the constructed interference ad-hoc selective family has the size $O((1+r(\mathcal{F}))\cdot((1+\log(\Delta_{max}/\Delta_{min})))\cdot\log|\mathcal{F}|)$. □

4 Centralized Broadcasting and Interference Ad-Hoc Selective Families

Now, we sketch a simple algorithm producing a radio broadcast schedule for radio networks modelled by arbitrary IRGs.

The algorithm works as follows. At first, we split the network nodes into layers $L_0, L_1, \ldots, L_{ecc}$ with respect to their distances to a fixed source s, where $L_i = \{v \in V | dist(s,v) = i\}$ and ecc is the eccentricity of the source s in the transmission subgraph. The source message is disseminated in phases layer by layer. During the i-th phase, the source message is received by the nodes of the layer L_i owing to transmissions of the nodes in the layer L_{i-1}. Particularly, for each node v of L_i, we construct a set-pair (T_v, I_v) such that $T_v = \{w \in L_{i-1} | (w,v) \in E_T\}$ and $I_v = \{w \in L_{i-1} | (w,v) \in E_I\}$. Furthermore, for a collection $\mathcal{F}^i = \{(T_v, I_v) | v \in L_i\}$, an interference ad-hoc selective family $\mathcal{S}^i = \{S_1^i, \ldots, S_m^i\}$ is obtained as an output of the algorithm described in the proof of the theorem 2. Finally, transmissions of the phase i are scheduled in such a way, that in the j-th round of the phase i exactly the nodes $S_j^i \subseteq L_{i-1}$ transmit the source message. Thus the i-th phase takes totally $|\mathcal{S}^i|$ rounds.

Theorem 3. *Let $G = (V, E_T \cup E_I)$ to be an IRG. There is a deterministic polynomial time algorithm that for a given source node s produces a schedule of the length $O((1+\log(\Delta_{max}/\Delta_{min}))\cdot R(s))$, where $R(s) = \sum_{i=0}^{ecc-1}((1+r(\mathcal{F}^i))\cdot \log|\mathcal{F}^i|)$ with ecc standing for the eccentricity of the node s in the transmission subgraph of G.*

Note, that there are IRGs for which the utilized layer by layer information dissemination approach is not suitable. For instance, consider the following undirected IRG $G_m = (V, E_T \cup E_I)$, where

- $V = \{s, v_1, \ldots, v_m, w_1, \ldots, w_m\}$
- $E_T = \{(s, v_i), (v_i, w_i) | i = 1, \ldots, m\} \cup \{(v_i, v_j), (w_i, w_j) | 1 \leq i \neq j \leq m\}$
- $E_I = \{(v_i, w_j) | 1 \leq i \neq j \leq m\}$

Observe, that the ratio of the incident interference edges to the incident transmission edges is at most 1, for each node of the constructed graph G_m. It is easy to see, that it is not possible to complete the broadcasting task from the node s in less then $m+1$ rounds utilizing the layer-by-layer approach. Indeed, all nodes of the layer $L_1 = \{v_1, \ldots, v_m\}$ have to transmit in separate rounds. On the other hand, broadcasting with the source s can be completed in 3 rounds:

1. the source s transmits and informs all nodes of the layer L_1
2. the node v_1 transmits and informs the node w_1
3. the node w_1 transmits and informs the remaining nodes.

Although this simple algorithm is not suitable for all IRGs, its combination with some graph analysis or heuristics can lead to algorithms that produce radio broadcast schedules of sufficient length (for practical applications), at least for a large subclass of radio networks.

5 Time Complexity of the Centralized Broadcasting in IRG with Respect to the Maximum Degree

Let Δ to be the maximum degree of IRG, i.e. the largest degree (the total number of incident, transmission and interference, edges of a node) over all networks nodes. In this section, we shall investigate the impact of the parameter Δ to the time complexity of the centralized broadcasting in IRGs.

Theorem 4. *Let $G = (V_S \cup V_R, E_T \cup E_I)$ to be a directed bipartite IRG, where E_T are the transmission edges and E_I are the interference edges. Suppose that all nodes in V_S are informed, i.e. they possess the source message, and the nodes in V_R are uninformed. Let Δ to be the maximum degree in the IRG G, i.e. $\Delta = max\{deg_T(v) + deg_I(v) | v \in V_S \cup V_R\}$, where $deg_T(v) = |\{(u,v)|(u,v) \in E_T \vee (v,u) \in E_T\}|$ and $deg_I(v) = |\{(u,v)|(u,v) \in E_I \vee (v,u) \in E_I\}|$. If $deg_T(v) \geq 1$ for all $v \in V_R$, then the following holds:*

- *all nodes in V_R can be informed in at most Δ^2 rounds,*
- *if $deg(v) = 1$ for all $v \in V_S$, then all nodes in V_R can be informed in at most $2 \cdot \Delta$ rounds.*

Proof. At first, we prove that the first part of the claim holds. Assignment of transmission rounds for the nodes of the set V_S can be obtained by a simple greedy algorithm *Greedy-rounds-assignment* (Algorithm 2). The goal of the algorithm is to assign a non-colliding round number $r(v) \in \{1, \ldots, \Delta^2\}$ to each node v. Two invariants are valid during the computation of assignments:

- For the nodes of the set V_S, the number $r(v)$ denotes a round, in which the node v transmits the source message.

Input : $G = (V_S \cup V_R, E_T \cup E_I)$ - bipartite IRG, nodes in V_S are informed, nodes in V_R are uninformed
Output: round assignment $r : (V_S \cup V_R) \longrightarrow \{1, \dots, \Delta^2\}$
initially, $r(v)$ is unassigned for each $v \in V_S \cup V_R$;
$A_S \leftarrow \emptyset$;
$A_R \leftarrow \emptyset$;
while $A_S \neq V_S$ **do**
  pick a random node v from $V_S \setminus A_S$;
  $B_I \leftarrow \{r(u) | u \in A_R \wedge (v, u) \in E_T \cup E_I\}$;
  $N \leftarrow \{u | u \in V_R \setminus A_R \wedge (v, u) \in E_T\}$;
  $B_N \leftarrow \{r(u) | u \in A_S \wedge \exists w \in N, (u, w) \in E_I \cup E_T)\}$;
  $B \leftarrow B_I \cup B_N$;
  $r(v) \leftarrow$ any element of the set $\{1, \dots, \Delta^2\} \setminus B$;
  $A_S \leftarrow A_S \cup \{v\}$;
  foreach $u \in V_R \setminus A_R$ *such that* $(v, u) \in E_T$ **do**
    $r(u) \leftarrow r(v)$;
    $A_R \leftarrow A_R \cup \{u\}$;
  end
end
return *assignment r*

Algorithm 2. Algorithm *Greedy-rounds-assignment*

– For the nodes of the set V_R, the value $r(v)$ denotes a round, in which the node v receives the source message.

Note that a node $v \in V_R$ can receive the source message also in other rounds than $r(v)$. The algorithm GRA produces an assignment r as a result of the following computation. In each iteration, we pick a node $v \in V_S$ such that $r(v)$ is unassigned. The set variables A_S and A_R contain only the nodes of the sets V_S and V_R, respectively. Each node w, that is a member of V_S or V_R, has already defined the value $r(w)$. For the picked node v, a set of colliding transmission rounds B is computed. The set B contains the round numbers of all nodes with assigned round number that are in the transmission or interference range of the node v. These round numbers are colliding due to the invariant property defined for the nodes in the set V_R. Particularly, the round number means a round when the source message is received for sure. Moreover, in order to inform all uninformed nodes in the transmission range of the picked node v, i.e. the nodes of set variable N with unassigned round numbers, we have to guaranty that none of their (transmission or interference) neighbors transmits in the round $r(v)$. This achieved by adding their round numbers to the set of colliding rounds B. The picked node v is a neighbor of at most Δ other nodes. Each of them adds to the set B at most $\Delta - 1$ colliding round numbers. Hence, it holds that $|B| \leq \Delta(\Delta - 1)$, and we can pick a non-colliding round number from the set $\{1, \dots, \Delta^2\} \setminus B$. It is easy to see, that if each node $v \in V_S$ transmits the source message in the round $r(v)$ then all nodes in V_R become informed in at most Δ^2 rounds. It concludes the proof of the first part of the claim.

Now we prove the second part of the claim. Observe, that if $deg_T(v) = 1$ for all $v \in V_S$, then in the algorithm GRA the set B of colliding round number contains at most $2 \cdot \Delta - 1$ rounds. Thus, we can modify the algorithm GRA in such a way that assigned round number is picked from the set $\{1, \ldots, 2 \cdot \Delta\} \setminus B$. Finally, transmissions according to the computed assignment ensure that all nodes in V_R become informed in at most $2 \cdot \Delta$ rounds. □

In [5], Gąsieniec et al. discussed centralized communication in radio networks (assuming the standard model without extended interference). They presented algorithms that produce a radio broadcast schedule of the length $O(D + \Delta \cdot \log n)$ and $D + O(\log^3 n)$, for a reachability graph G. Algorithms are based on the gathering spanning tree and can be reformulated as follows:

Theorem 5. *Let $G' = (V_S \cup V_R, E)$ to be an undirected bipartite graph. Suppose that all nodes in V_S are informed (possess the source message) and the nodes in V_R are uninformed. Let $n = |V_S \cup V_R|$ and denote as*

- *$A_S(n)$ the maximal length of a schedule produced by an algorithm A_S ensuring that all nodes in V_R become informed due to transmissions of the nodes in V_S,*
- *$A_F(n)$ the maximal length of a schedule produced by an algorithm A_F ensuring that all nodes in the set V_R become informed due to transmissions of the nodes in V_S under the following assumption: $deg(v) = 1$, for all $v \in V_S \cup V_R$.*

Let $G = (V, E)$ to be a reachability graph (no extended interference). There is an algorithm (schema) that produces a radio broadcast schedule of the length $O(A_F(n) \cdot D + A_S(n) \cdot \log n)$, where $n = |V|$.

Now, we show how to realize centralized broadcasting in radio networks (with extended interference) modelled by an IRG.

Theorem 6. *Let $G = (V, E_T \cup E_I)$ to be an IRG with the maximum degree Δ. There is a deterministic polynomial time algorithm that for a given source node s produces a radio broadcast schedule of the length $O(\Delta D + min\{\Delta, \log \Delta \cdot \log n\} \cdot \Delta \log n)$.*

Proof. We utilize the schema of the theorem 5 for the transmission subgraph of a given IRG G. Due to the presence of interference edges, we cannot apply algorithms for the standard "non-interference" model. However observe, that the algorithm presented in the second part of proof of the theorem 4 produces a schedule such that $A_F(n) = O(\Delta)$. Moreover, the algorithm presented in the first part of the proof can be used as the algorithm A_S, in order to produce schedules such that $A_S(n) = O(\Delta^2)$. Another choice for the algorithm A_S is to apply the algorithm presented in the theorem 3 that leads to a schedule such that $A_S(n) = O(\Delta \cdot \log \Delta \cdot \log n)$. Note that this algorithm provides shorter schedules than the former algorithm in the case when $\Delta = \Omega(\log^2 n)$. □

One can easily show that it is possible to construct an IRG with the maximum degree Δ such that the broadcasting time is lower-bounded by $\Omega(\Delta \cdot D)$ rounds.

6 Conclusion

In this paper, we introduced a new model (an extension of the standard graph model) of radio networks that reflects a situation when a transmission of a node causes interference in an area where the decoding of this transmission is impossible. We focused on the broadcasting problem in the newly proposed model. Designed algorithms, one of them based on the introduced notion of interference ad-hoc selective families, can be seen as a first step to study the efficiency of communication in this model.

The evident open problem is design of optimal communication (broadcasting, gossiping, etc.) algorithms with respect to parameters of an underlying IRG that express the presence of interference edges in an appropriate way. This could answer the question how the presence of interference edges makes the communication process more difficult (e.g. slower) in compare to the communication under the standard graph model.

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A Appendix

Lemma 1. *The conditional probabilities $Y_j(S,(T,I))$ and $N_j(S,(T,I))$ can be computed in $O(n)$ time.*

Proof. Evaluation of the conditional probabilities $Y_j(S,(T,I))$ and $N_j(S,(T,I))$ is based on the following equalities. In these equalities, we use α to denote $|\delta_j(T)| + |\delta_j(I)|$.

– $\delta_j(T) = 0$

$$Y_j(S,(T,I)) = N_j(S,(T,I)) = \begin{cases} \left(1-\frac{1}{\Delta}\right)^{|\delta_j(I)|} & |T\cap S| = 1 \wedge I \cap S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

– $\delta_j(T) \geq 1$

- $u_j \in T$

$$Y_j(S,(T,I)) = \begin{cases} \left(1-\frac{1}{\Delta}\right)^{\alpha-1} & (T\cup I)\cap S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$N_j(S,(T,I)) = \begin{cases} 0 & |(T\cup I)\cap S| \geq 2 \\ 0 & |T\cap S| = 0 \wedge |I\cap S| = 1 \\ \left(1-\frac{1}{\Delta}\right)^{\alpha-1} & |T\cap S| = 1 \wedge |I\cap S| = 0 \\ 0 & |(T\cup I)\cap S| = 0 \wedge |\delta_j(T)| = 1 \\ (|\delta_j(T)|-1)\cdot\frac{1}{\Delta}\cdot\left(1-\frac{1}{\Delta}\right)^{\alpha-2} & |(T\cup I)\cap S| = 0 \wedge |\delta_j(T)| \geq 2 \end{cases}$$

- $u_j \in I$

$$Y_j(S,(T,I)) = 0$$

$$N_j(S,(T,I)) = \begin{cases} 0 & |(T\cup I)\cap S| \geq 2 \\ 0 & |T\cap S| = 0 \wedge |I\cap S| = 1 \\ \left(1-\frac{1}{\Delta}\right)^{\alpha-1} & |T\cap S| = 1 \wedge |I\cap S| = 0 \\ |\delta_j(T)|\cdot\frac{1}{\Delta}\cdot\left(1-\frac{1}{\Delta}\right)^{\alpha-2} & |(T\cup I)\cap S| = 0 \end{cases}$$

- $u_j \notin T \cup I$

$$\begin{matrix} Y_j(S,(T,I)) \\ N_j(S,(T,I)) \end{matrix} = \begin{cases} 0 & |(T\cup I)\cap S| \geq 2 \\ 0 & |T\cap S| = 0 \wedge |I\cap S| = 1 \\ \left(1-\frac{1}{\Delta}\right)^{\alpha} & |T\cap S| = 1 \wedge |I\cap S| = 0 \\ |\delta_j(T)|\cdot\frac{1}{\Delta}\cdot\left(1-\frac{1}{\Delta}\right)^{\alpha-1} & |(T\cup I)\cap S| = 0 \end{cases}$$

□